5.5 Undetermined Coeffs (continued)

example $y'' - y = \cosh(x)$

$\cosh(x) = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$

$\sinh(x) = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x}$

$\frac{d}{dx}\cosh(x) = \sinh(x)$

$\frac{d}{dx}\sinh(x) = \cosh(x)$

$y = y_c + y_p$

$y_c \to$ homogeneous only

$y_p \to$ due to right side

$y_c = c_1 e^{x} + c_2 e^{-x}$ (solution to $y'' - y = 0$)

for y_p, there are two options

option 1: rewrite $\cosh(x)$ as exponentials

$$y'' - y = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$$

$$y_c = c_1 e^{x} + c_2 e^{-x}$$

y_p: guess a form based on right side

$$\frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$$

guess: $Ae^{x} + Be^{-x}$

check if y_p is duplicating y_c

here, both terms of y_p are duplicating y_c

fix: throw an x at y_p

$$y_p = Axe^{x} + Bxe^{-x}$$

now sub y_p into $y'' - y = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$

$$y_p = Axe^{x} + Bxe^{-x}$$

$$y_p' = Ae^{x} + Axe^{x} + Be^{-x} - Bxe^{-x}$$

$$y_p'' = 2Ae^{x} + Axe^{x} - 2Be^{-x} + Bxe^{x}$$

after subbing in and equating like terms

$$A = \frac{1}{4}$$

$$B = -\frac{1}{4}$$

so, $y_p = \frac{1}{4}xe^{x} - \frac{1}{4}xe^{-x}$

and $\boxed{y = c_1 e^{x} + c_2 e^{-x} + \frac{1}{4}xe^{x} - \frac{1}{4}xe^{-x}}$

option 2: work w/ hyperbolic functions

$$y'' - y = \cosh(x)$$

$$y_c = c_1 e^{x} + c_2 e^{-x}$$

rewrite using hyperbolic functions

$$\cosh(x) = \tfrac{1}{2}e^{x} + \tfrac{1}{2}e^{-x}$$

$$\sinh(x) = \tfrac{1}{2}e^{x} - \tfrac{1}{2}e^{-x}$$

$$e^{x} = \cosh(x) + \sinh(x)$$

$$e^{-x} = \cosh(x) - \sinh(x)$$

$$y_c = c_1(\cosh(x) + \sinh(x)) + c_2(\cosh(x) - \sinh(x))$$

$$= \underbrace{(c_1 + c_2)}_{\text{"}c_1\text{"}}\cosh(x) + \underbrace{(c_1 - c_2)}_{\text{"}c_2\text{"}}\sinh(x)$$

$$y_c = c_1 \cosh(x) + c_2 \sinh(x)$$

right side: $\cosh(x)$

guess y_p: $y_p = A\cosh(x) + B\sinh(x)$

ALWAYS include BOTH $\cosh(x)$ and $\sinh(x)$
just like w/ $\cos(x)$ and $\sin(x)$

check duplication: there is $\cosh(x)$ and $\sinh(x)$
in y_c and y_p

fix: multiply by x

$$y_p = Ax\cosh(x) + Bx\sinh(x)$$

$$y_p' = Ax\sinh(x) + A\cosh(x) + B\sinh(x) + Bx\cosh(x)$$

$$y_p'' = 2A\sinh(x) + Ax\cosh(x) + Bx\sinh(x) + 2B\cosh(x)$$

sub into $y'' - y = \cosh(x)$, equating like terms

⋮

$A = 0$, $B = \frac{1}{2}$ $\qquad y_p = \frac{1}{2}x\sinh(x)$

$$\boxed{y = C_1\cosh(x) + C_2\sinh(x) + \frac{1}{2}x\sinh(x)}$$

<u>example</u> $y^{(5)} + 2y^{(4)} - y = 3$

find y_p

guess y_p : right side is 3

so, guess $y_p = A$

check duplication: is there a constant in y_c ?

$y^{(5)} + 2y^{(4)} - y = 0$

char. eq : $r^5 + 2r^4 - 1 = 0$

how to factor ??

but, if there were to be a constant in y_c,

one root has to be $r = 0 \rightarrow e^{0x} = 1 \rightarrow c \cdot 1$

does $r = 0$ work in $r^5 + 2r^4 - 1 = 0$?

NO, so $r = 0$ is NOT a root

so, there is no duplication.

so, $y_p = A$ is ok.

now sub it into $y^{(5)} + 2y^{(4)} - y = 3$

$-A = 3$

so, $A = -3$

so, $\boxed{y_p = -3}$

example Find the form of y_p: $y'' + y = x\cos x$

$y_c = c_1 \cos x + c_2 \sin x$

y_p: guess using form of $x\cos x$

$x\cos x = \underbrace{(x+0)}_{\text{guess } Ax+B \text{ (1st-deg)}} \underbrace{\cos x}_{\text{guess } \cos(x) \text{ AND } \sin(x)}$

so, $y_p = (Ax+B)\cos x + (Cx+D)\sin x$

$= Ax\cos x + B\cos x + Cx\sin x + D\sin x$

check for duplication: $\cos(x)$, $\sin(x)$ in both y_c and y_p

fix: multiply $\cos(x)$ and $\sin(x)$ in y_p by x

but then they duplicate $x\cos x$ and $x\sin x$ in y_p

fix: multiply $\cos(x)$ and $\sin(x)$ in y_p by x again

(so, mult. by x^2)

correct y_p: $y_p = Ax\cos x + B\underline{\underline{x}}^2\cos x + Cx\sin x + D\underline{\underline{x}}^2\sin x$

or, multiply the initial guess of y_p by x

$y_p = Ax^2\cos x + Bx\cos x + Cx^2\sin x + Dx\sin x$

same form

example $y^{(5)} + 2y^{(3)} + 2y'' = 3x^2 - 1$

form of y_p ?

initial guess of y_p : $y_p = Ax^2 + Bx + C$ (2nd deg)

check duplication: $r^5 + 2r^3 + 2r^2 = 0$

$$r^2 \underbrace{(r^3 + 2r + 2)}_{\text{not } 0} = 0$$

$r = 0, 0$

so, y_c has : $y_c = c_1 + c_2 x + \dots$

$y_p = Ax^2 + Bx + C$

two duplication

fix: multiply entire y_p by as many factors of x until no more duplication

$y_p = x(Ax^2 + Bx + C)$ still duplicating parts of y_c

fix: another x

$\boxed{y_p = x^2(Ax^2 + Bx + C)}$ now no more duplication!