5.5 Nonhomogeneous Eqs. : The Variation of Parameters

$y'' + ay' + by = f(x) \rightarrow y = y_c + y_p$

undetermined coeffs : y_p has same form as $f(x)$

↳ limitations: y_p form must not change after differentiation (e.g. exponential remains exponential)

→ good w/ polynomials, exponentials and sine / cosine

but if $f(x) = \underbrace{\tan x}$ $f'(x) = \sec^2 x$ changed form!

undetermined coeff cannot handle this

likewise, we can't use undetermined coeffs on $\ln x$

today, we look at the <u>variation of parameters</u>

$$y'' + ay' + by = f(x)$$

$$y_c = c_1 y_1 + c_2 y_2$$

y_1 and y_2 ~~are~~ form the complementary solution

the general solution, instead of $y = y_c + y_p$, we express it as

$$\boxed{y = \underbrace{u_1(x)}_{\text{"parameters"}} y_1 + \underbrace{u_2(x)}_{} y_2}$$

goal: find u_1 and u_2.

sub into $y'' + ay' + by = f(x)$

$$\left\{\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ y' &= u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2 \\ y' &= u_1 y_1' + u_2 y_2' \\ y'' &= u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2' \end{aligned}\right.$$

we will set

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2' + a u_1 y_1' + a u_2 y_2' + b u_1 y_1 + b u_2 y_2 = f(x)$$

$$u_1 \underbrace{(y_1'' + a y_1' + b y_1)}_{0} + u_2 \underbrace{(y_2'' + a y_2' + b y_2)}_{0} + u_1' y_1' + u_2' y_2' = f(x)$$

because y_1 is part of y_c

y_c is solution to $y'' + ay' + by = 0$

same reason

we are left w/ $u_1' y_1' + u_2' y_2' = f(x)$

together w/ the condition we set earlier: $u_1' y_1 + u_2' y_2 = 0$

so, we solve $\boxed{\begin{array}{l} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = f(x) \end{array}}$ for u_1', u_2' then integrate

then, general solution is $y = u_1 y_1 + u_2 y_2$

<u>example</u> $y'' - y' - 2y = 3e^{2x}$

first, find y_c : $y_c = c_1 e^{2x} + c_2 e^{-x}$

$y_1 = e^{2x} \quad y_2 = e^{-x}$

if we were to use undetermined coeffs, $y_p = Axe^{2x}$
(variation of parameters doesn't require this adjustment)

← because there is e^{2x} in y_c

solve $u_1' y_1 + u_2' y_2 = 0$

$u_1' y_1' + u_2' y_2' = 3e^{2x}$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 3e^{2x} \end{bmatrix}$$

$$\left[\begin{array}{cc|c} y_1 & y_2 & 0 \\ y_1' & y_2' & 3e^{2x} \end{array}\right] = \left[\begin{array}{cc|c} e^{2x} & e^{-x} & 0 \\ 2e^{2x} & -e^{-x} & 3e^{2x} \end{array}\right]$$

$$\xrightarrow{-2R_1+R_2} \begin{bmatrix} e^{2x} & e^{-x} & 0 \\ 0 & -3e^{-x} & 3e^{2x} \end{bmatrix}$$

row 2 : $-3e^{-x} u_2' = 3e^{2x}$

$$u_2' = -e^{3x}$$

row 1 : $e^{2x} u_1' + e^{-x} u_2' = 0$

$$e^{2x} u_1' - e^{2x} = 0$$

$$u_1' = 1$$

integrate : $u_1 = \int 1 \, dx = x + c_1$

$$u_2 = \int -e^{3x} dx = -\frac{1}{3} e^{3x} + c_2$$

general solution : $y = u_1 y_1 + u_2 y_2$

$$y = (x + c_1)\, e^{2x} + \left(-\tfrac{1}{3} e^{3x} + c_2\right) e^{-x}$$

$$= c_1 e^{2x} + c_2 e^{-x} + x e^{2x} \underbrace{- \tfrac{1}{3} e^{2x}}_{\text{combine}}$$

$$\boxed{y = \underbrace{c_1 e^{2x} + c_2 e^{-x}}_{y_c} + \underbrace{x e^{2x}}_{y_p}}$$

example $\quad y'' + y = \sec x$ $\quad$ undetermined coeffs cannot handle this

first find y_c: $\quad r^2+1=0 \quad r = \pm i$

$$y_c = c_1 \cos x + c_2 \sin x$$

$$y_1 = \cos x \qquad y_2 = \sin x$$

solve $\quad u_1' y_1 + u_2' y_2 = 0$

$$u_1' y_1' + u_2' y_2' = f(x)$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u_1' \\ u_2' \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ \sec x \end{bmatrix}}_{\vec{b}}$$

$$\vec{x} = A^{-1} \vec{b}$$

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \sec x \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \underbrace{\frac{1}{\boxed{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}}} \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}}_{\text{inverse of } \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}} \begin{bmatrix} 0 \\ \sec x \end{bmatrix}$$

Wronskian of y_1, y_2

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} -\sin x \sec x \\ 1 \end{bmatrix}$$

$$u_1 = \int -\sin x \cdot \frac{1}{\cos x}\, dx$$

$$u_2 = \int 1 \, dx$$

$$u_1 = \ln|\cos x| + c_1 \qquad u_2 = x + c_2$$

$$y = u_1 y_1 + u_2 y_2$$

$$\boxed{y = \underbrace{c_1 \cos x + c_2 \sin x}_{y_c} + \underbrace{\cos x \ln|\cos x| + x \sin x}_{y_p}}$$