5.5 Nonhomogeneous Eqs. – Variation of Parameters

$y'' + ay' + by = f(x) \longrightarrow y = y_c + y_p$

solution to $y'' + ay' + by = 0$

from $f(x)$

undetermined coeffs: y_p has the same form as $f(x)$

→ we can only handle $f(x)$ such that it doesn't change form after differentiation

polynomial remains polynomial after deriv.

exponential " exponential " "

sine, cosine " sine, cosine " "

but others are troublesome

for example, $f(x) = \tan x$

if we guess $y_p = A \tan x$

$y_p' = A \sec^2 x$ not tangent anymore

the method of variation of parameters doesn't have that limitation plus it automatically takes care of the duplication issue

Variation of Parameters

$y'' + ay' + by = f(x)$

"parameters"

$\rightarrow$ $y = y_c + y_p$ but it is also $\boxed{y = u_1(x)\, y_1 + u_2(x)\, y_2}$

where y_1, y_2 are the solutions of $y'' + ay' + by = 0$

$(y_c = c_1 y_1 + c_2 y_2)$

find u_1 and u_2 : sub $y = u_1 y_1 + u_2 y_2$ into $y'' + ay' + by = f(x)$

$y_p' = u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2$

$y' = u_1 y_1' + u_2 y_2'$

$y'' = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2'$

to avoid having to deal w/ u_1'' and u_2'', we set $\boxed{u_1' y_1 + u_2' y_2 = 0}$

Sub into $y'' + ay' + by = f(x)$

$$u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2' + a u_1 y_1' + a u_2 y_2' + b u_1 y_1 + b u_2 y_2 = f(x)$$

$$u_1 \underbrace{(y_1'' + a y_1' + b y_1)}_{\substack{0 \text{ because} \\ y_1 \text{ is a solution} \\ \text{to } y'' + ay' + by = 0}} + u_2 \underbrace{(y_2'' + a y_2' + b y_2)}_{\substack{0 \text{ for the} \\ \text{same reason}}} + u_1' y_1' + u_2' y_2' = f(x)$$

so, $u_1' y_1' + u_2' y_2' = f(x)$

together w/ $u_1' y_1 + u_2' y_2 = 0$

we get this system :

$$\boxed{\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= f(x) \\ &\text{or} \\ \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} &= \begin{bmatrix} 0 \\ f(x) \end{bmatrix} \end{aligned}}$$

Solve for u_1', u_2' then integrate, then

$$y = u_1 y_1 + u_2 y_2$$

<u>example</u> $y'' - y' - 2y = 3e^{2x}$

y_c: $r^2 - r - 2 = 0$

$(r-2)(r+1) = 0 \quad r_1 = 2 \quad r_2 = -1$

so, $y_c = c_1 e^{2x} + c_2 e^{-x}$

$y_1 = e^{2x} \qquad y_2 = e^{-x}$

(if we were to use undetermined coeffs, the duplication in e^{2x} requires $y_p = A\underline{\underline{x}}e^{2x}$)

but variation does not care

Solve: $u_1' y_1 + u_2' y_2 = 0$

$u_1' y_1' + u_2' y_2' = f(x)$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$\begin{bmatrix} e^{2x} & e^{-x} & 0 \\ 2e^{2x} & -e^{-x} & 3e^{2x} \end{bmatrix}$$ do row reduction

$$\xrightarrow{-2R_1+R_2} \begin{array}{c} \begin{array}{cc} u_1' & u_2' \end{array} \\ \left[\begin{array}{ccc} e^{2x} & e^{-x} & 0 \\ 0 & -3e^{-x} & 3e^{2x} \end{array}\right] \end{array}$$

$$R_2: \quad -3e^{-x}u_2' = 3e^{2x}$$

$$u_2' = -e^{3x} \rightarrow \boxed{u_2 = -\tfrac{1}{3}e^{3x} + c_2}$$

$$R_1: \quad e^{2x}u_1' + e^{-x}u_2' = 0$$

$$e^{2x}u_1' + e^{-x}(-e^{3x}) = 0$$

$$e^{2x}u_1' = e^{2x}$$

$$u_1' = 1 \rightarrow \boxed{u_1 = x + c_1}$$

general solution: $y = u_1 y_1 + u_2 y_2$

$$= (x+c_1)e^{2x} + \left(-\tfrac{1}{3}e^{3x} + c_2\right)e^{-x}$$

$$= \underbrace{c_1 e^{2x}} + c_2 e^{-x} + xe^{2x} \underbrace{- \tfrac{1}{3}e^{2x}}$$

absorb

$$= \boxed{\underbrace{c_1 e^{2x} + c_2 e^{-x}}_{y_c} + \underbrace{xe^{2x}}_{y_p}}$$

example

$y'' + y = \sec x$

y_c: $y_c = c_1 \cos x + c_2 \sin x$

$y_1 = \cos x \quad y_2 = \sin x$

Solve:
$$\underbrace{\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u_1' \\ u_2' \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ f(x) \end{bmatrix}}_{\vec{b}}$$

$$A\vec{x} = \vec{b}$$
$$\vec{x} = A^{-1}\vec{b}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$= \frac{1}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

Wronskian of y_1 and y_2
($\neq 0$ because y_1, y_2 are indp)

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{W(x)} \begin{bmatrix} -y_2 f(x) \\ y_1 f(x) \end{bmatrix} \qquad W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$u_1' = \frac{-y_2 f(x)}{W(x)} \qquad u_1 = \int \frac{-y_2 f(x)}{W(x)}\,dx$$

$$u_2' = \frac{y_1 f(x)}{W(x)} \qquad u_2 = \int \frac{y_1 f(x)}{W(x)}\,dx$$

here, $y_1 = \cos x \quad y_2 = \sin x \quad f(x) = \sec x$

$$u_1 = \int -\sin x \cdot \frac{1}{\cos x}\,dx = \cdots = \ln|\cos x| + C_1$$

skip C_1, C_2 if only want y_p

$$u_2 = \int \cos x \cdot \frac{1}{\cos x}\,dx = x + C_2$$

gen. solution: $y = u_1 y_1 + u_2 y_2 = (\ln|\cos x| + C_1)\cos x + (x + C_2)\sin x$

$$\boxed{y = \underbrace{C_1 \cos x + C_2 \sin x}_{y_c} + \underbrace{\cos x \ln|\cos x| + x \sin x}_{y_p}}$$