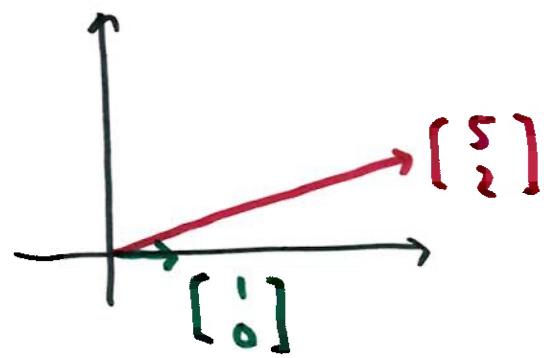
6.1 Eigenvalues and Eigenvectors

$$A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$$

when multiply a vector by this matrix, the vector usually changes direction and magnitude

for example, $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad A\vec{x} = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$



$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad A\vec{x} = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

most matrices have special vectors that preserve their directions after transformation

$A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad A\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ direction preserved
same magnitude

$\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad A\vec{x} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

same direction
magnitude increased by factor of 5

these vectors that keep their directions are called the eigenvectors of the matrix

the factors by which their magnitudes change are called the eigenvalues so, in the above example,

$\begin{bmatrix}0\\1\end{bmatrix}$ is an eigenvector w/ the associated eigenvalue of 1

$\begin{bmatrix}2\\1\end{bmatrix}$ " " " " 5

How to find them?

if $\vec{v}$ is an eigenvector, then $A\vec{v}$ keeps the same direction by changes magnitude by a factor of λ (~~lamda~~ lambda)

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} - \lambda\vec{v} = \vec{0}$$

$$\boxed{(A - \lambda I)\vec{v} = \vec{0}}$$

I: identity matrix

this is a homogeneous eq, which always has solution $\vec{v} = \vec{0}$ (trivial solution) but we don't want that

to force nontrivial solution, we force multiple solutions which implies $\boxed{|A-\lambda I| = \det(A-\lambda I) = 0}$

so, to find eigenvalues and eigenvectors, we solve

$\det(A-\lambda I) = 0$ for λ's

then use those λ's to solve for $\vec{v}$'s from

$(A-\lambda I)\vec{v} = \vec{0}$

example

$$A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

first, find eigenvalues

$$\det(A - \lambda I) = \begin{vmatrix} 7-\lambda & 4 \\ -3 & -1-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)(-1-\lambda) - (-3)(4) = 0$$

$$\vdots$$

$$\boxed{\lambda^2 - 6\lambda + 5 = 0}$$ characteristic eq.

this is an n^{th}-order polynomial for an $n \times n$ matrix

$$(\lambda - 5)(\lambda - 1) = 0$$

$$\boxed{\lambda = 1, \quad \lambda = 5}$$ eigenvalues of A

now find the associated eigenvectors

solve $(A-\lambda I)\vec{v} = \vec{0}$ for each λ

$\underline{\lambda = 1}$ $(A-\lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 6 & 4 \\ -3 & -2 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 4 & 0 \\ -3 & -2 & 0 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & \end{array} \\ \begin{bmatrix} 3 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

there is ALWAYS at least one zero row

$$\vec{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$x_2 = r$ (free)

$x_1 \longrightarrow$ row 1 : $3x_1 + 2x_2 = 0$

$x_1 = -\frac{2}{3}x_2 = -\frac{2}{3}r$

so, $\vec{v} = \begin{bmatrix} -\frac{2}{3}r \\ r \end{bmatrix} = r\begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$

choose any convenient r $(\neq 0)$

here, let's use $r = -3$

so, $\boxed{\vec{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \lambda = 1}$

next pair

$\underline{\lambda = 5}$ solve $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 2 & 4 & 0 \\ -3 & -6 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\vec{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $x_2 = r$

$x_1 + 2x_2 = 0$ $x_1 = -2x_2 = -2r$

$\vec{v} = \begin{bmatrix} -2r \\ r \end{bmatrix}$ choose $r = 1$

$$\boxed{\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad \lambda = 5}$$

<u>example</u>

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Solve $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = 0$$

Cofactor expansion along column 2

$$(1-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(4-\lambda)(1-\lambda) - (-2)(1)\right] = 0$$

$$(1-\lambda)(\lambda^2 - 5\lambda + 6) = 0$$

$$(1-\lambda)(\lambda-2)(\lambda-3) = 0$$

$$\boxed{\lambda = 1, 2, 3}$$

$3\times3 \to 3$ λ's

solve $(A-\lambda I)\vec{v}=\vec{0}$ for each λ

<u>$\lambda=1$</u> $(A-\lambda I)\vec{v}=\vec{0}$

$$\begin{bmatrix} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \vec{v}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$x_2 = r$ row 2: $x_3 = 0$

row 1: $x_1 = 0$ $\vec{v}=\begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix}$

choose $r=1$

$$\boxed{\vec{v}=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \lambda=1}$$

this eigenvector spans an eigenspace of dimension 1

$\underline{\lambda=2}$ $(A-\lambda I)\vec{v}=\vec{0}$ ... $\vec{v}=\begin{bmatrix}-1\\2\\2\end{bmatrix}$

$\underline{\lambda=3}$ ... $\vec{v}=\begin{bmatrix}-1\\1\\1\end{bmatrix}$

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if A is triangular or diagonal, its <u>eigenvalues</u> are the <u>main diagonal elements</u>

$A=\begin{bmatrix}1 & 2 & 3\\0 & 4 & 5\\0 & 0 & 6\end{bmatrix}$ triangular

$\det(A-\lambda I)=0$ $\begin{vmatrix}1-\lambda & 2 & 3\\0 & 4-\lambda & 5\\0 & 0 & 6-\lambda\end{vmatrix}=0$

$$(1-\lambda)\begin{vmatrix} 4-\lambda & 5 \\ 0 & 6-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(4-\lambda)(6-\lambda) = 0 \rightarrow \lambda = 1, 4, 6$$

(main diagonal elements)

eigenvalues can be complex → come in conjugate pairs

$$A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \qquad \begin{vmatrix} 0-\lambda & 1 \\ -4 & 0-\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 4 = 0 \qquad \boxed{\lambda = 2i, -2i}$$

eigenvectors are also complex conjugate pairs

⋮

$$\vec{v} = \begin{bmatrix} -1 \\ -2i \end{bmatrix}, \lambda = 2i \qquad \vec{v} = \begin{bmatrix} -1 \\ 2i \end{bmatrix}, \lambda = -2i$$