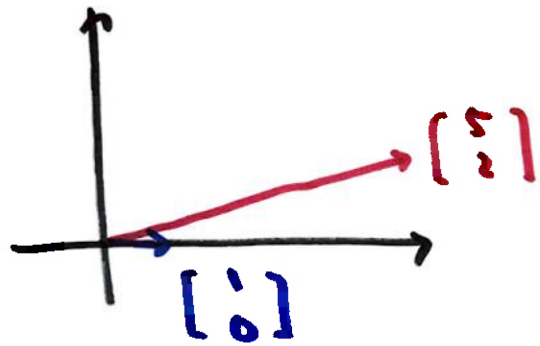
6.1 Eigenvalues and Eigenvectors

$A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$ when multiplied by a vector, the matrix usually changes the direction and magnitude of the vector

for example, $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad A\vec{x} = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$



$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad A\vec{x} = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

but, there are some special vectors that preserve their directions for example, $\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $A\vec{x} = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $A\vec{x} = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

these are examples of special vectors that retain their directions after transformation

→ called <u>eigenvectors</u>

(each $n \times n$ matrix has n of these)

they may change their lengths, the factor by which the length changes is called an <u>eigenvalue</u>

so, $A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$ has these eigenvalue/eigenvector

lambda

pairs: $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\lambda = 1$

$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda = 5$

How to find these eigenvalue/eigenvector pairs?

if $\vec{v}$ is an eigenvector of A with the associated eigenvalue of λ, then

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} - \lambda\vec{v} = \vec{0}$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

identity w/ same size as A

to find λ's, we need to force a nontrivial solution in $(A-\lambda I)\vec{v} = \vec{0}$ → $\vec{v} \neq \vec{0}$

so we want non unique/infinitely-many solutions

→ $\boxed{\det(A - \lambda I) = 0}$ solve for λ's

then use the λ's we found to solve

$\boxed{(A - \lambda I)\vec{v} = \vec{0}}$ for the $\vec{v}$'s example

$$A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$$

first, solve $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7-\lambda & 4 \\ -3 & -1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 7-\lambda & 4 \\ -3 & -1-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)(-1-\lambda) + 12 = 0$$

$$\boxed{\lambda^2 - 6\lambda + 5 = 0}$$ characteristic equation

$$(\lambda - 5)(\lambda - 1) = 0$$

$$\boxed{\lambda = 1, \quad \lambda = 5}$$ eigenvalues (n of these for an n×n matrix)

now solve $(A-\lambda I)\vec{v}=\vec{0}$ using the λ's we found

$\underline{\lambda=1}$ $\quad (A-\lambda I)\vec{v}=\vec{0}$

$$\begin{bmatrix} 7-\lambda & 4 \\ -3 & -1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \vec{v}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 4 & 0 \\ -3 & -2 & 0 \end{bmatrix}$$

$$\rightarrow \cdots \rightarrow \begin{bmatrix} 3 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

there is <u>ALWAYS</u> at least one row of zeros

$x_2=r$

$3x_1+2x_2=0 \qquad x_1=-\frac{2}{3}r$

choose ANY convenient $r\neq 0$

$\vec{v}=\begin{bmatrix} -\frac{2}{3}r \\ r \end{bmatrix}$ choose $r=-3$ $\quad \boxed{\vec{v}=\begin{bmatrix} 2 \\ -3 \end{bmatrix} \quad \lambda=1}$

repeat for $\underline{\lambda = 5}$

$$\begin{bmatrix} 2 & 4 & 0 \\ -3 & -6 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_2 = r$ $\quad$ $x_1 + 2x_2 = 0$

$x_1 = -2r$

$\vec{v} = \begin{bmatrix} -2r \\ r \end{bmatrix}$ choose $r = 1$ $\boxed{\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \lambda = 5}$

there are infinitely-many choice or r, so there are infinitely-many eigenvectors, but they all are on the same line

→ ~~the~~ all the eigenvectors are in a one-dimensional eigenspace spanned by $\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ (basis vector)

<u>example</u>

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

start with $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = 0$$

expand along 2nd column

$$(1-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(4-\lambda)(1-\lambda) + 2\right] = 0$$

$$(1-\lambda)(\lambda^2 - 5\lambda + 6) = 0$$

$$(1-\lambda)(\lambda-2)(\lambda-3) = 0 \qquad \lambda = 1, 2, 3$$

solve $(A-\lambda I)\vec{v}=\vec{0}$ for each λ

$\underline{\lambda=1}$:
$$\begin{bmatrix} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_2=r \quad x_3=0 \quad x_1=0$

$\vec{v}=\begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix}$ choose $r=6$ $\boxed{\vec{v}=\begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}, \lambda=1}$

$\underline{\lambda=2}$: $\vec{v}=\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

$\underline{\lambda=3}$: $\vec{v}=\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

eigenvalues for triangular or diagonal matrix is easy to find : main diagonal elements

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \qquad \det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{vmatrix} = 0$$

$$(1-\lambda) \begin{vmatrix} 4-\lambda & 5 \\ 0 & 6-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(4-\lambda)(6-\lambda) = 0$$

$$\lambda = 1,\ 4,\ 6$$

eigenvalues can be complex

<u>example</u> $A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$

$$\det(A-\lambda I) = 0 = \begin{vmatrix} -\lambda & 1 \\ -4 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 4 = 0 \qquad \boxed{\lambda = 2i, -2i}$$

conjugate pairs

solve $(A-\lambda I)\vec{v} = \vec{0}$

<u>$\lambda = 2i$</u> $\begin{bmatrix} -2i & 1 & 0 \\ -4 & -2i & 0 \end{bmatrix}$ $(-2i)(2i) = -4i^2 = 4$

$$\xrightarrow{(2i)R_1 + R_2} \begin{bmatrix} -2i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

still at least one row of zeros

$x_2 = r \qquad -2i\,x_1 + x_2 = 0$

$$x_1 = \frac{1}{2i} r$$

$$\vec{v} = \begin{bmatrix} \frac{1}{2i} r \\ r \end{bmatrix} \qquad \text{choose } r = 2i$$

$$\boxed{\vec{v} = \begin{bmatrix} 1 \\ 2i \end{bmatrix} \qquad \lambda = 2i}$$

$\underline{\lambda = -2i}$ $\quad \ldots \quad$ $\boxed{\vec{v} = \begin{bmatrix} 1 \\ -2i \end{bmatrix} \qquad \lambda = -2i}$

nullity : dimension of $\underbrace{\text{null space}}_{\text{solution space to } A\vec{x} = \vec{0}}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \hline 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \end{array}$$

$$\rightarrow \begin{array}{cccc} & & x_3 & \\ \boxed{1} & 2 & 3 & 0 \\ 0 & \boxed{-3} & -6 & 0 \end{array}$$

2 pivots

x_3 is free $\quad x_3 = r$

$-3x_2 - 6x_3 = 0$

$x_2 = -2r$

$$x_1 + 2x_2 + 3x_3 = 0$$

$$x_1 = -2x_2 - 3x_3$$

$$= 4r - 3r = r$$

$$\vec{x} = \begin{bmatrix} r \\ -2r \\ r \end{bmatrix} = r \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

null space dimension 1

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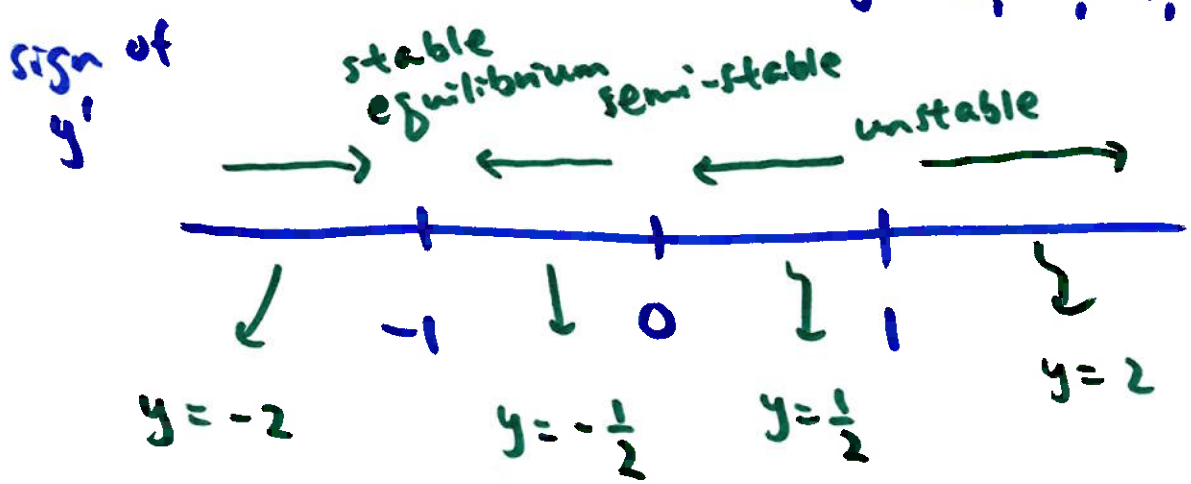
$$y' = y^2(y^2 - 1)$$

critical pts: $y' = 0$ $\quad y^2 = 0, \quad y^2 - 1 = 0$

$$y = 0, 0, 1, -1$$



phase diagram

space containing all vectors of the form

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{such that} \quad x = 2z$$

is this a subspace?

Subspace: existence of $\vec{0}$?

closure under addition

closure under scalar multiplication

$$\vec{x} = \begin{bmatrix} 2z \\ y \\ z \end{bmatrix} \quad \text{1st = twice of last}$$

$\vec{0}$? yes (choose $y=0$, $z=0$)

choose two such vectors, is their sum still in the space (1st = twice of last)

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -4 \\ 6 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ -1 \end{bmatrix}$$

want to see this for ALL

$$\begin{bmatrix} 2z \\ y \\ z \end{bmatrix}$$

one vector: $\begin{bmatrix} 2a \\ b \\ a \end{bmatrix}$

other vector: $\begin{bmatrix} 2c \\ d \\ c \end{bmatrix}$

add them: $\begin{bmatrix} 2a+2c \\ b+d \\ a+c \end{bmatrix} = \begin{bmatrix} 2(a+c) \\ b+d \\ a+c \end{bmatrix}$

shows closure under addition