7.2 Matrices and Linear Systems

n^{th}-order $\longleftrightarrow$ n 1st-order eqs. in a system

for example, $t^2 x'' + t x' + (t^2-1)x = e^t$

$$x'' + \frac{1}{t} x' + \frac{t^2-1}{t^2} x = \frac{e^t}{t^2}$$

let $z_1 = x$

$z_2 = x'$

eq 1: $z_1' = z_2$

eq 2: from the diff. eq

$$x'' = -\frac{1}{t} x' - \frac{t^2-1}{t^2} x + \frac{e^t}{t^2}$$

$$z_2' = -\frac{1}{t} z_2 - \frac{t^2-1}{t^2} z_1 + \frac{e^t}{t^2}$$

$$z_1' = z_2$$

$$z_2' = -\frac{t^2-1}{t^2} z_1 - \frac{1}{t} z_2 + \frac{e^t}{t^2}$$

as a matrix eg:

$$\underbrace{\begin{bmatrix} z_1' \\ z_2' \end{bmatrix}}_{\vec{x}\,'} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{t^2-1}{t^2} & -\frac{1}{t} \end{bmatrix}}_{p(t)} \underbrace{\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} 0 \\ \frac{e^t}{t^2} \end{bmatrix}}_{\vec{f}(t)}$$

in the form $\vec{x}\,' = P(t)\vec{x} + \vec{f}(t)$

$P(t)$ → coefficient matrix

$\vec{f}(t)$ → nonhomogeneous term

turn into a matrix eq:

$$x' = 8x - 2y + z + t$$
$$y' = x - 4z + t^2$$
$$z' = 5y - 2z + t^3$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 8 & -2 & 1 \\ 1 & 0 & -4 \\ 0 & 5 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}$$

$$\vec{x}\,' = P(t)\vec{x} + \vec{f}(t)$$

basic properties are similar to a first-order scalar eq.

if $\vec{f}(t) = \vec{0} \rightarrow \vec{x}\,' = P(t)\vec{x}$ is called a

<u>homogeneous system</u>

the solution $\vec{x}$ is a vector that satisfies the matrix eq. $\vec{x}' = P(t)\vec{x} + \vec{f}(t)$

if $P(t)$ is $n \times n$ $\rightarrow$ n solutions $\vec{x}_1, \vec{x}_2, \vec{x}_3, \dots, \vec{x}_n$

(because $n \times n \rightarrow n$ 1st-order eqs in a system

$\leftrightarrow$ one n^{th}-order scalar eq.)

linear combination of the solutions is also a solution

the general solution to the homogeneous system is

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \cdots + c_n \vec{x}_n$$

these solutions are linearly independent

$$c_1 \vec{x}_1 + c_2 \vec{x}_2 + \cdots + c_n \vec{x}_n = \vec{0} \rightarrow c_1 = c_2 = \cdots = c_n = 0$$

the <u>Wronskian</u> of these solutions is

$$W(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n) = \begin{vmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \cdots & \vec{x}_n \\ & & & & \end{vmatrix}$$

(solutions as columns)

if $W \neq 0$ on an interval of t, then the solutions are linearly indp on that interval.

<u>example</u> Verify that $\vec{x}_1 = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{x}_2 = e^{-2t}\begin{bmatrix} 1 \\ 5 \end{bmatrix}$

are solutions of $\vec{x}\,' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\vec{x}$

and show that they are linearly indp.

if they are the solutions, then they satisfy the differential eq.

$$\vec{x}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \vec{x}$$

$$\vec{x}_1 = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ sub into system above

$$\vec{x}_1' = 2e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$2e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \left(e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

$$= e^{2t} \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$2e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{2t} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$ same, so $e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a solution

we can similarly verify that $\vec{x}_2 = e^{-2t}\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ is also a solution

let's look at the Wronskian of $\vec{x}_1 = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{x}_2 = e^{-2t}\begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$$W(\vec{x}_1, \vec{x}_2) = \begin{array}{c} \begin{vmatrix} e^{2t} & e^{-2t} \\ e^{2t} & 5e^{-2t} \end{vmatrix} \\ \vec{x}_1 \qquad \vec{x}_2 \end{array}$$

$$= e^{2t} \cdot 5e^{-2t} - e^{2t} \cdot e^{-2t}$$

$$= 5 - 1 = 4 \neq 0 \text{ for any } t \rightarrow \vec{x}_1 \text{ and } \vec{x}_2$$

indp on $-\infty < t < \infty$

$$\vec{x}\,' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \vec{x} \qquad \vec{x}_1 = e^{\boxed{2}t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{x}_2 = e^{\boxed{-2}t} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

let's find the eigenvalues and eigenvectors of the coefficient matrix

$$A = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 \\ 5 & -3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-3-\lambda) + 5 = 0$$

$$-9 - 3\lambda + 3\lambda + \lambda^2 + 5 = 0 \rightarrow \lambda^2 - 4 = 0$$

$$\cancel{\lambda^2 - 4 = 0} \qquad \lambda = \boxed{2}, \boxed{-2}$$

eigenvectors

$\underline{\lambda = 2}$ solve $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 1 & -1 & 0 \\ 5 & -5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_2 = r \qquad x_1 = x_2 = r$

$\vec{v} = r \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ choose $r = 1$

$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{x}_1 = e^{\boxed{2}t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

eigenvalue

corresponding eigenvector

$\vec{x}_2$ is found the same way

$$\vec{x}' = P(t)\vec{x} + \vec{f}(t)$$

if $\vec{f}(t) \not\equiv \vec{0}$, then the nonhomogeneous system has solution

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

$\vec{x}_p$ → particular solution (due to presence of $\vec{f}(t)$)

$\vec{x}_c$ → complementary solution

→ solution to

$$\vec{x}' = P(t)\vec{x}$$

(no $\vec{f}(t)$)