7.3 The Eigenvalue Method for Linear Systems

Solve constant coefficient homogeneous systems

$$\vec{x}' = A\vec{x} \qquad A: \text{constant matrix}$$

first graphical interpretation of solutions

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

pick x_1, x_2

the system gives

vector tangent to solution

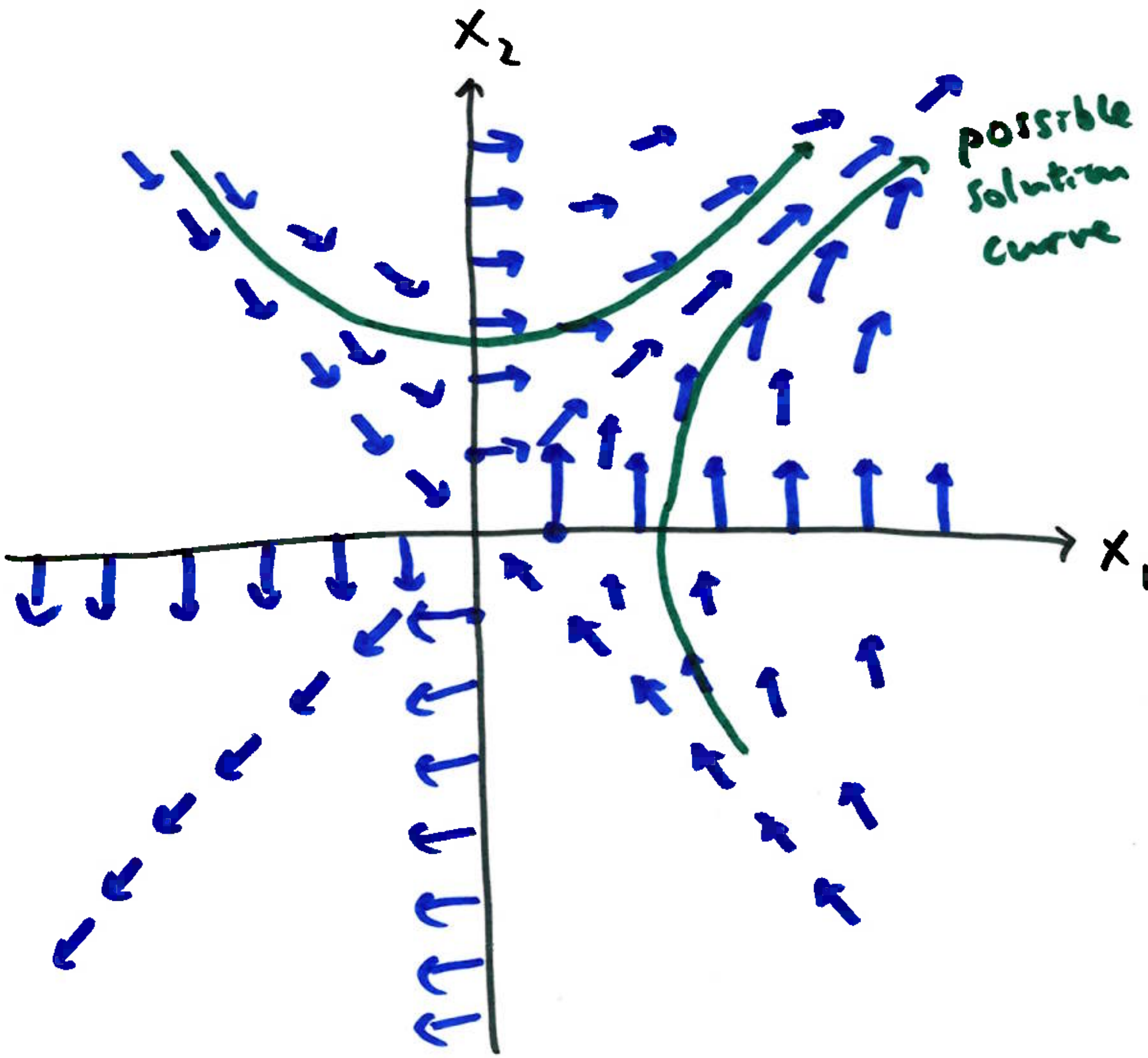
at x_1, x_2 we picked

for example, if we picked $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $\vec{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{x}' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$\vec{x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $\vec{x}' = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

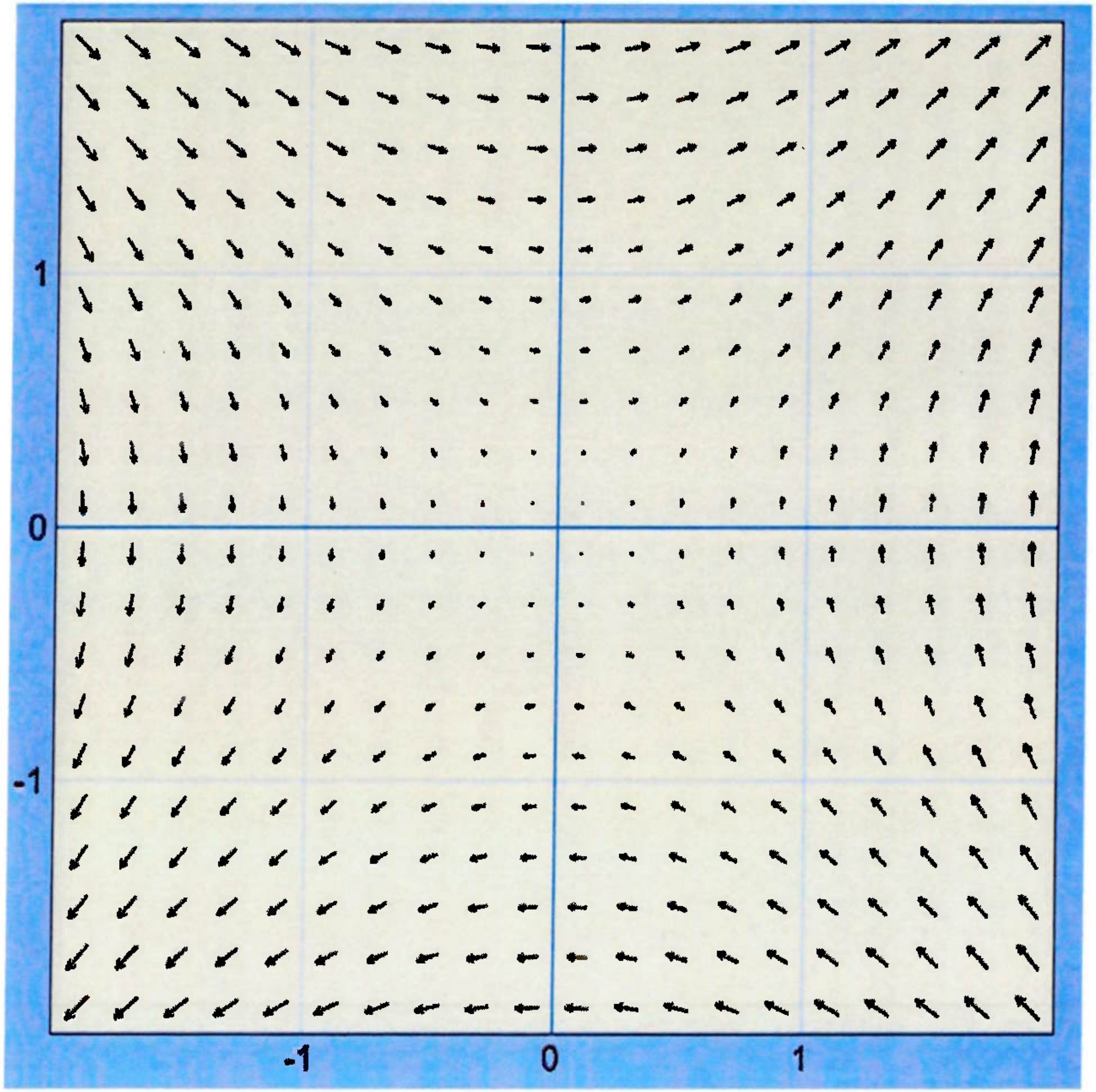
$\vec{x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $\vec{x}' = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$



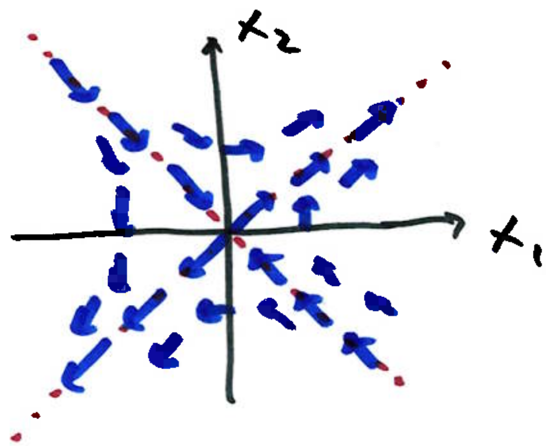
repeat w/ other points to fill up the space

$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{x}' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

these are vectors tangent to <u>solution curves</u>


1
0
-1
-1
0
1

notice we see two straight line solutions



they are : $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

these can be found from the matrix A w/o plotting the arrows

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x} \qquad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

eigenvalues and eigenvectors of A :

$\det(A - \lambda I) = 0 \qquad \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \qquad \lambda^2 = 1 \quad \lambda = 1, -1$

$\underline{\lambda = 1}$ solve $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \therefore \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

this is one of the straight line solutions

its eigenvalue $\lambda = 1$ is positive so the arrows go away from origin

$\underline{\lambda = -1}$ solve $(A - \lambda I)\vec{v} = \vec{0}$

$$\vdots$$

$$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda < 0$ so they go into the origin

the actual solution to $\vec{x}\,' = A\vec{x}$ is found this way (as shown last time)

$$\boxed{\vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2}$$

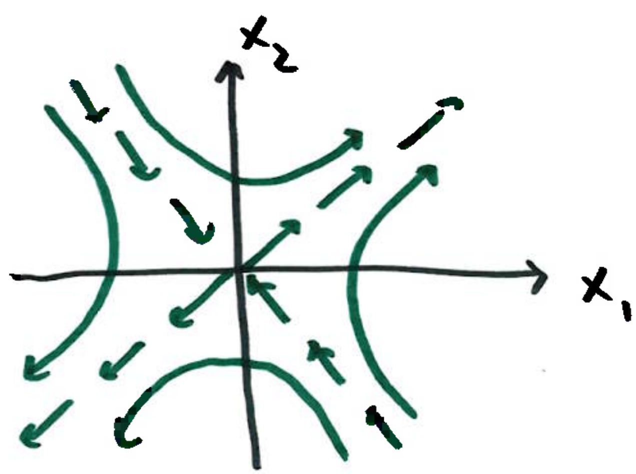
$\lambda_1, \vec{v}_1$ and $\lambda_2, \vec{v}_2$ are the eigenvalue/eigenvector pairs for this system, $\vec{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x}$

solution is $\vec{x} = c_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

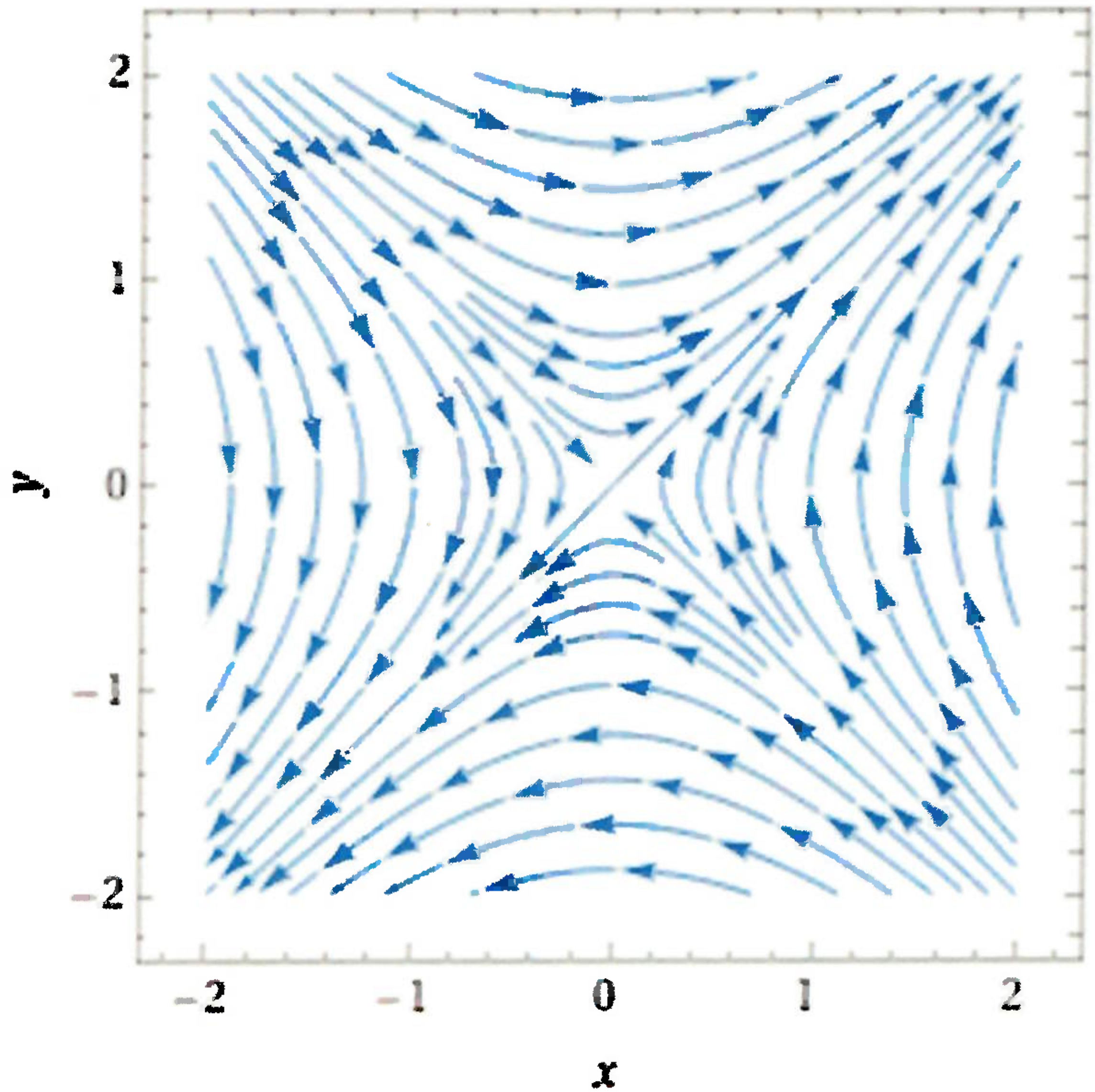
the straight line solutions show up :

$\lim_{t \to \infty} \vec{x} \Rightarrow e^{-t} \to 0$ so solutions end up following $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lim_{t \to -\infty} \vec{x} \to e^{t} \to 0$ so solutions follow $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$



phase portrait



2
1
0
−1
−2
y
−2
−1
0
1
2
x

example $\vec{x}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{x} \longleftrightarrow \begin{array}{l} x_1' = x_1 + 2x_2 \\ x_2' = 2x_1 + x_2 \end{array}$

Solution: $\vec{x} = c_1 e^{\lambda_1 t} \vec{v_1} + c_2 e^{\lambda_2 t} \vec{v_2}$

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 - 4 = 0$$

$$(1-\lambda)^2 = 4$$

$$1-\lambda = \pm 2$$

$$\lambda = -1, \quad \lambda = 3$$

$\lambda = -1$: straight line for this goes into origin

$\lambda = 3$: goes away from origin

$\underline{\lambda = -1}$ Solve $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\underline{\lambda = 3}$

$$\begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

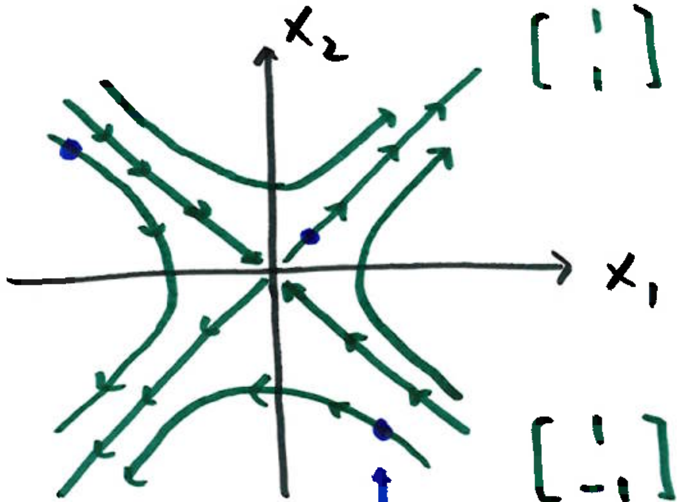
$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solution: $\vec{x} = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

1st row: $x_1(t) = c_1 e^{-t} + c_2 e^{3t}$

2nd row: $x_2(t) = -c_1 e^{-t} + c_2 e^{3t}$

quick sketch of phase portrait



$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda > 0$

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \lambda < 0$

solution if
this is the initial condition

example

$$x_1' = 4x_1 + 3x_2 + 3x_3$$
$$x_2' = -8x_1 - 7x_2 - 3x_3$$
$$x_3' = 8x_1 + 8x_2 + 4x_3$$

$$\downarrow$$

$$\vec{x}' = \begin{bmatrix} 4 & 3 & 3 \\ -8 & -7 & -3 \\ 8 & 8 & 4 \end{bmatrix} \vec{x}$$

A is 3×3 $\rightarrow$ 3 eigenvalue / eigenvector pairs

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 4-\lambda & 3 & 3 \\ -8 & -7-\lambda & -3 \\ 8 & 8 & 4-\lambda \end{vmatrix} = 0 \quad \cdots \quad \lambda = -4, 4, 1$$

corresponding eigenvectors:

$$\vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$(\lambda = -4)$ $(\lambda = 4)$ $(\lambda = 1)$

solution is the same: linear combo of $e^{\lambda t}\vec{v}$

$$\vec{x} = c_1 e^{-4t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

this is how we solve $\vec{x}' = A\vec{x}$ if λ's are real and distinct

next time: complex λ's