7.3 The Eigenvalue Method (Continued)

$\vec{x}' = A\vec{x}$ if A is constant matrix, $n \times n$

then there are n eigenvalue/eigenvector pairs

$\lambda_1, \lambda_2, \ldots, \lambda_n$

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$

n solutions: $e^{\lambda_i t}\vec{v}_i$

general solution: linear combo of them

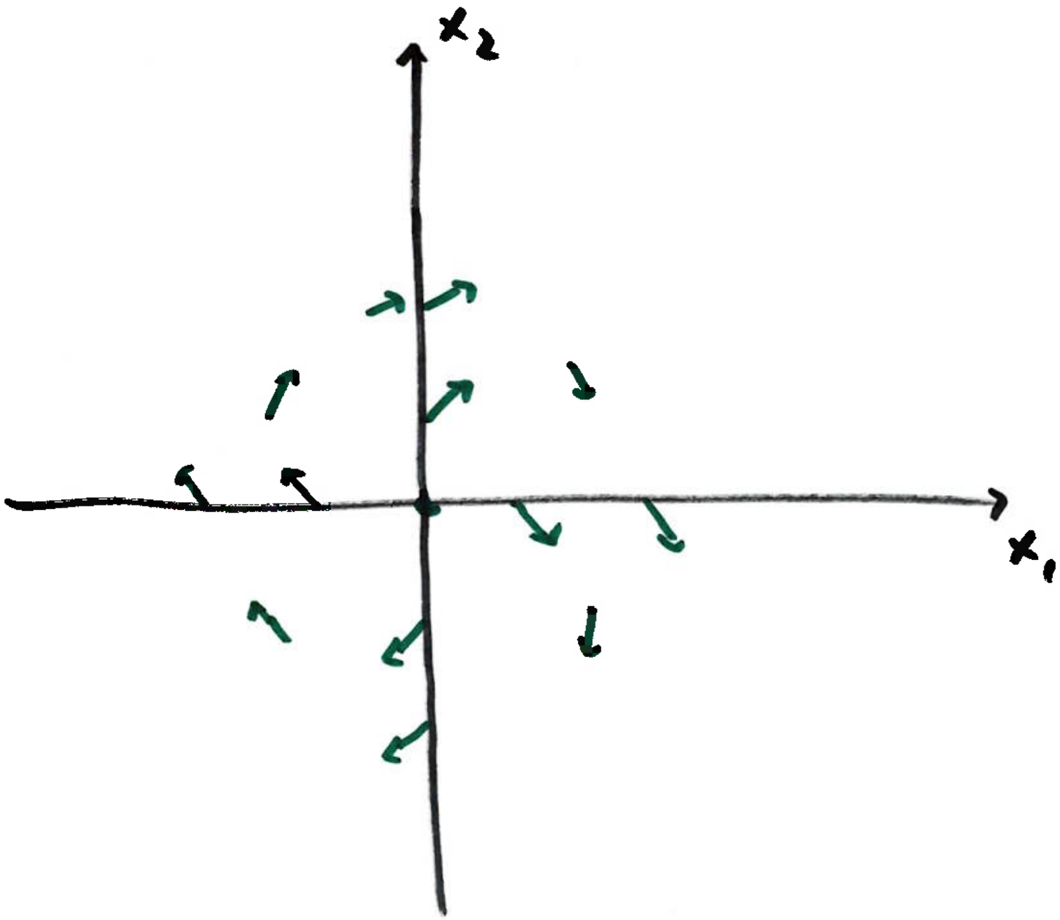
$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \cdots + c_n e^{\lambda_n t}\vec{v}_n$$

last time: real, distinct λ's

today: complex λ's

let's start w/ a slope field

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}$$

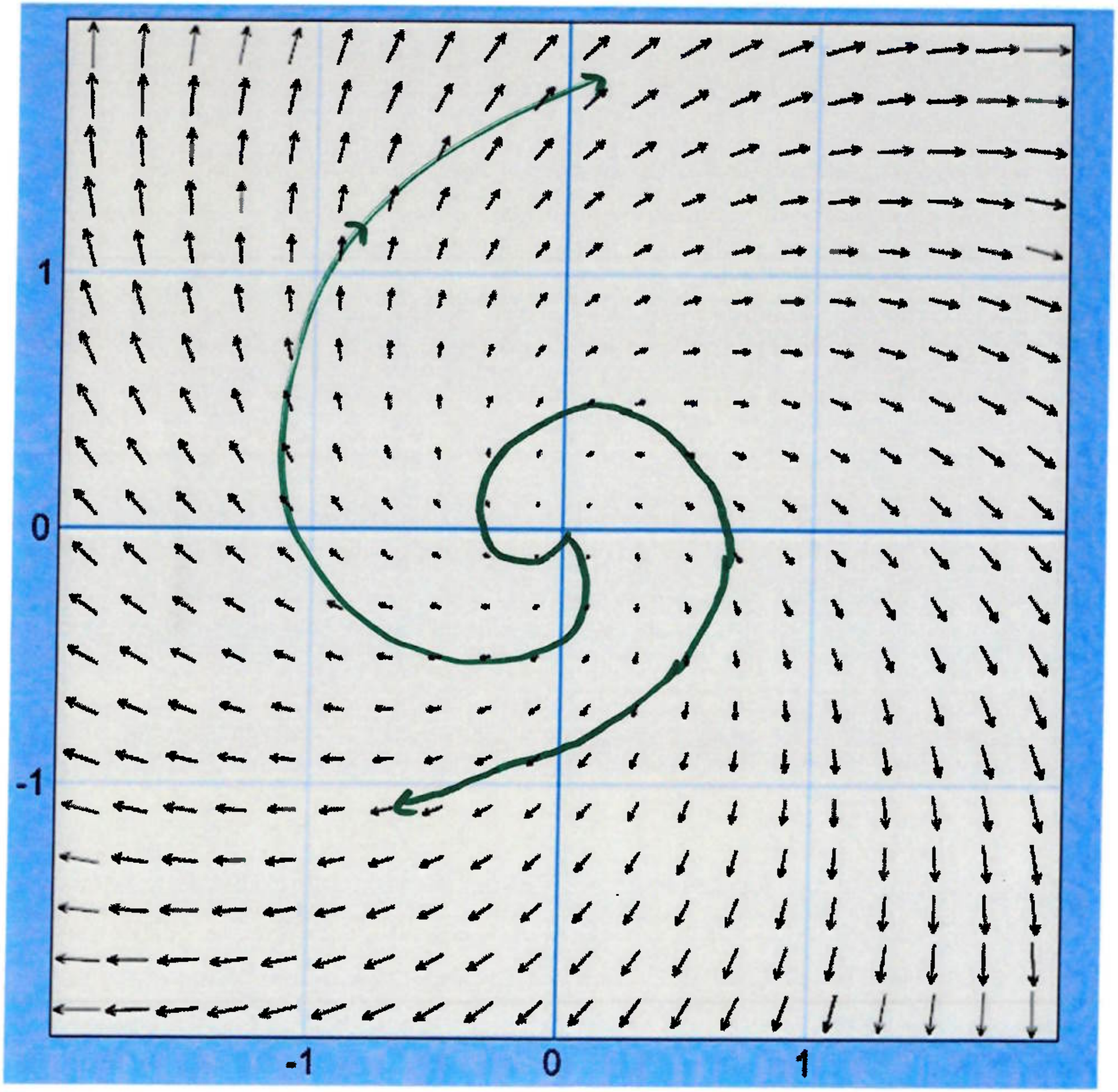


$\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\vec{x}' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{x}' = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\vec{x}' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vdots$



complex
eigenvalues
↓
spiral
direction
field

$$\vec{x}' = \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{A} \vec{x}$$

solutions are still $e^{\lambda_i t} \vec{v}_i$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 + 1 = 0$$

$$(1-\lambda)^2 = -1$$

$$1-\lambda = \pm i$$

$$\boxed{\lambda = 1+i, \quad 1-i}$$

always conjugate pairs

eigenvectors:

$\boxed{\underline{\lambda = 1+i}}$ $\quad (A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} -i & 1 & 0 \\ -1 & -i & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & i & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \boxed{\vec{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}}$$

$\boxed{\underline{\lambda = 1-i}}$ $\begin{bmatrix} i & 1 & 0 \\ -1 & i & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\boxed{\vec{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}}$

$\vec{v}$'s are also in conjugate pairs

two solutions in the form $e^{\lambda t}\vec{v}$

$\lambda = 1+i, \; \vec{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$e^{\lambda t}\vec{v} = e^{(1+i)t}\begin{bmatrix} 1 \\ i \end{bmatrix}$

Euler's formula: $e^{it} = \cos t + i \sin t$

$= e^{t} e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} = e^{t}(\cos t + i \sin t)\begin{bmatrix} 1 \\ i \end{bmatrix}$

$= e^{t}\begin{bmatrix} \cos t + i \sin t \\ -\sin t + i \cos t \end{bmatrix} = \underbrace{e^{t}\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}_{\text{real part}} + i\, \underbrace{e^{t}\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}_{\text{imaginary part}}$

repeat for $\lambda = 1-i$, $\vec{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$$e^{(1-i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix} = e^{t} e^{i(-t)} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$= e^{t}(\cos t - i \sin t)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$= e^{t}\begin{bmatrix} \cos t - i \sin t \\ -\sin t - i \cos t \end{bmatrix}$$

$$= \underbrace{e^{t}\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}_{\text{real part}} - i \underbrace{e^{t}\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}_{\text{imaginary part}}$$

again, complex conjugate pairs.

complex λ's: one solution is $\vec{u} + i\vec{v}$ and the other is $\vec{u} - i\vec{v}$

just like w/ scaler equations, we use the real and imaginary parts of either solution as the fundamental solutions for the general solution.

one solution: $\vec{u} + i\vec{v}$

general solution: $c_1 \vec{u} + c_2 \vec{v}$ (no i)

here, the general solution to $\vec{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}$ is

$$\boxed{\vec{x}(t) = c_1 e^t \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}$$

complex λ's $\rightarrow$ spirals $\rightarrow$ cosines + sines there is a complication: how we choose eigenvector

<u>example</u>

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{x}$$

$$\lambda = \pm i$$

work out <u>one</u> solution, identify real and imaginary parts

let's choose $\underline{\lambda = -i}$

solve $(A - \lambda I)\vec{v} = 0$

$$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ b is free: $b = r$ $a = ib = ir$

$\vec{v} = \begin{bmatrix} ir \\ r \end{bmatrix} = r \begin{bmatrix} i \\ 1 \end{bmatrix}$ choose ANY $r \neq 0$

if $r = -i$, $\vec{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

if $r = 1$, $\vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$

let's work out the solution using these choices

start w/ $\lambda = -i$, $\vec{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$$e^{\lambda t}\vec{v} = e^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix} = (\cos t - i\sin t)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$= \begin{bmatrix} \cos t - i\sin t \\ -\sin t - i\cos t \end{bmatrix} = \underbrace{\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}_{\vec{u}} - i\underbrace{\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}_{\vec{v}}$$

general solution: $\boxed{\vec{x} = c_1\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}$

repeat w/ $\lambda = -i$, $\vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$$e^{\lambda t}\vec{v} = e^{-it}\begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$= (\cos t - i\sin t)\begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} i\cos t + \sin t \\ \cos t - i\sin t \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}_{\vec{u}} + i\underbrace{\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}_{\vec{v}}$$

general solution: $\boxed{\vec{x} = c_1\begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + c_2\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}$

these two solutions are the same even though they look different example

$$\vec{x}' = \begin{bmatrix} 3 & 0 & 1 \\ 9 & -1 & 2 \\ -9 & 4 & -1 \end{bmatrix} \vec{x}$$

possibilities: 3 real λ's (distinct or repeated)

next time

2 complex and one real

$\lambda = 3, \; -1+i, \; -1-i$

$$\vec{v} = \begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} -4-i \\ -9+2i \\ 17 \end{bmatrix}, \begin{bmatrix} -4+i \\ -9-2i \\ 17 \end{bmatrix}$$

general solution is formed the same way

$$e^{3t}\begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix} \rightarrow \lambda = 3, \; \vec{v} = \begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix}$$

then work out either $e^{(-1+i)t}\begin{bmatrix} -4-i \\ -9+2i \\ 17 \end{bmatrix}$

or $e^{(-1-i)t}\begin{bmatrix} -4+i \\ -9-2i \\ 17 \end{bmatrix}$ and get the real and imag parts