7.6 Multiple/Repeated Eigenvalues

$\vec{x}' = Ax$ solutions are $e^{\lambda t}\vec{v}$

gen. solution $\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \cdots + c_n e^{\lambda_n t}\vec{v}_n$

$\lambda, \vec{v}$ pairs

no issues if λ's are distinct or complex

potential problems if λ's are repeated

$\vec{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$ $\lambda = 1, 1$

the eigenvalue of 1 is repeated twice (algebraic multiplicity of two)

eigenvector: $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$$

here, both a and b are free

let $a = r$, $b = s$

then $\vec{v} = \begin{bmatrix} r \\ s \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

form solutions using

general solution:

$$\boxed{\vec{x} = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

if matrix is complete, enough eigenvectors to form linearly indp solutions → solutions are formed the usual way.

the eigenspace has a dimension of two

→ two basis vectors (eigenvectors)

→ geometric multiplicity of two

if algebraic multiplicity = geo. multiplicity we say the matrix A is <u>complete</u>

now look at

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x} \qquad \lambda = 1, 1$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

only one free variable

$$\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} \qquad \begin{aligned} b &= 0 \\ a &= \text{free} = r \end{aligned}$$

$$\vec{v} = \begin{bmatrix} r \\ 0 \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

eigenspace dimension is <u>one</u>

only <u>one</u> eigenvector

we are missing a vector

first solution: $e^{\lambda_1 t}\vec{v}_1 = e^{t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

2nd one: $e^{\lambda t}\vec{v}_2$?

when we are missing eigenvectors, we say the matrix A is <u>defective</u>

in scalar case, if the roots of characteristic eq. are repeated, we multiply the existing solution by t to form a new one

$$y'' + 10y' + 25y = 0$$

$$r^2 + 10r + 25 = 0 \rightarrow r = 5, 5$$

$$y_1 = e^{5t} \qquad y_2 = t\, e^{5t}$$

so, we might expect we form the 2nd solution in $\vec{x}' = A\vec{x}$ by doing the same thing: $\vec{x}_1 = e^{\lambda t}\vec{v}_1$

$$\cancel{\vec{x}_2 = e^{\lambda t}\vec{v}_1 \underline{\underline{t}}}$$

BUT, this does NOT work

instead, we need $\boxed{\vec{x}_2 = e^{\lambda t}(t\vec{v}_1 + \vec{v}_2)}$

ordinary eigenvector

need to find (generalized eigenvector)

how to find $\vec{v}_2$?

$\vec{x}\,' = A\vec{x}$ eigenvalue λ only one eigenvector $\vec{v}_1$

the second solution is $\vec{x}_2 = e^{\lambda t}(t\vec{v}_1 + \vec{v}_2) = te^{\lambda t}\vec{v}_1 + e^{\lambda t}\vec{v}_2$

sub into $\vec{x}\,' = A\vec{x}$

$\vec{x}_2\,' = t\lambda e^{\lambda t}\vec{v}_1 + e^{\lambda t}\vec{v}_1 + \lambda e^{\lambda t}\vec{v}_2$

$t\lambda e^{\lambda t}\vec{v}_1 + e^{\lambda t}\vec{v}_1 + \lambda e^{\lambda t}\vec{v}_2 = At e^{\lambda t}\vec{v}_1 + Ae^{\lambda t}\vec{v}_2$

compare like terms

$te^{\lambda t}$ terms: $\lambda\vec{v}_1 = A\vec{v}_1 \rightarrow$ nothing new, definition of λ being and eigenvalue and $\vec{v}_1$ an eigenvector

$e^{\lambda t}$ terms: $\underbrace{\vec{v}_1 + \lambda\vec{v}_2 = A\vec{v}_2}_{\text{gives us } \vec{v}_2}$

$$\boxed{(A-\lambda I)\vec{v}_2 = \vec{v}_1}$$

also, from $(A-\lambda I)\vec{v}_2 = \vec{v}_1$

multiply by $A-\lambda I$ on both sides

$$(A-\lambda I)^2\vec{v}_2 = (A-\lambda I)\vec{v}_1 = \vec{0}$$ because $\vec{v}_1$ is eigenvector

So, there are two ways to find $\vec{v}_2$

$$(A-\lambda I)\vec{v}_2 = \vec{v}_1$$

or

$$(A-\lambda I)^2\vec{v}_2 = \vec{0}$$

revisit $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}$

$\lambda = 1, 1$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

1st solution: $e^{\lambda t}\vec{v}_1$

2nd " : $e^{\lambda t}(t\vec{v}_1 + \vec{v}_2)$

let's find $\vec{v}_2$ using $(A - \lambda I)\vec{v}_2 = \vec{v}_1$

$$\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$b = 1$ $a = r$ $\vec{v}_2 = \begin{bmatrix} r \\ 1 \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

choose ANY r

here, choose $r = 0$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

1st solution: $e^{t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

2nd solution: $e^{t}\left(t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

general solution:

$$\boxed{\vec{x} = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \left(t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)}$$

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second way to find $\vec{v}_2$: $(A-\lambda I)^2 \vec{v}_2 = \vec{0}$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \lambda = 1$$

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad (A-\lambda I)^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$(A-\lambda I)^2 \vec{v}_2 = \vec{0}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$$ $\vec{v}_2$ is almost arbitrary

choose ANY $\vec{v}_2$ EXCEPT $(A-\lambda I)\vec{v}_2 = \vec{0}$

basically, ANY $\vec{v}_2$ that is linearly indp from the true eigenvector $\vec{v}_1$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

choose ANY $\vec{v}_2$ indp from that

so, choose $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ but $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is ok, too

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for 3x3 matrix (or beyond 3x3), the second way is better

in general, if we are missing n-1 eigenvectors, then

$(A - \lambda I)^n$ = zero matrix

so, choose the missing one from $(A - \lambda I)^n \vec{v} = \vec{0}$

then build the rest using $(A - \lambda I)\vec{v}_n = \vec{v}_{n-1}$

example

$$\vec{x}' = \begin{bmatrix} 2 & 0 & 0 \\ -4 & 6 & 4 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}$$

$\lambda = 2, 2, 6$ $\qquad$ $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ for $\lambda = 6$

for $\lambda = 2$, $\quad (A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -4 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

two free variables
$\rightarrow$ two eigenvectors
(good, A is complete)

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

solution is formed normally

$$\vec{x} = c_1 e^{6t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$