7.6 Repeated Eigenvalues (part 1)

$\vec{x}' = A\vec{x}$ $\quad$ A: constant matrix

Solution: $\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \cdots + c_n e^{\lambda_n t}\vec{v}_n$

$\lambda_n, \vec{v}_n$: eigenvalue/eigenvector pairs of A

$n \times n$ A : n pairs

λ's distinct and complex are already covered.

if λ's are repeated things can get messy

Simplest case: for example

$$\vec{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$$

A is diagonal so λ's are the main diagonal elements

$\lambda = 1, \quad \lambda = 1$

$\lambda=1$ appears twice we say it has an algebraic multiplicity of two.

find eigenvectors : solve $(A-\lambda I)\vec{v}=\vec{0}$

$\underline{\lambda=1}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{v}=\begin{bmatrix} a \\ b \end{bmatrix}$$ both a and b are free

$$= a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

the eigenspace has dimension of two

there are two basis vectors, each can be used as an eigenvector

$$\vec{v_1}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v_2}=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

there are two eigenvectors

$\rightarrow$ geometric multiplicity is two

if algebraic multiplicity = geometric mult

we say the matrix A is <u>complete</u>

solutions are : $e^{\lambda_1 t}\vec{v_1} = e^{t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$e^{\lambda_1 t}\vec{v_2} = e^{t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

general solution $\vec{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

now let's look at $\vec{x}\,' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}$

triangular, so λ's are main diagonal elements

$\lambda = 1, \quad \lambda = 1$ alg. mult. is <u>two</u>

<u>$\lambda = 1$</u> $(A - \lambda I)\vec{v} = \vec{0}$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$

row 1: $b = 0$

a is free

$\vec{v} = \begin{bmatrix} a \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ dimension is one

geo. mult. is <u>one</u>

if geo.mult < alg. mult

(not enough eigenvectors)

we say the matrix A is <u>defective</u>

first solution: $e^{\lambda t}\vec{v} = e^{t}\begin{bmatrix}1\\0\end{bmatrix}$

second solution: ? (no $\vec{v_2}$)

with scalar case, for example $y'' + 10y' + 25y = 0$

char. eq. $r^2 + 10r + 25 = 0 \rightarrow r = 5, 5$ repeated

$$y_1 = e^{5t} \qquad y_2 = te^{5t}$$

so, it would be a good guess to find a second solution the same way $\vec{x_1} = e^{t}\begin{bmatrix}1\\0\end{bmatrix}$

$$\cancel{\vec{x_2} = te^{t}\begin{bmatrix}1\\0\end{bmatrix}}$$

BUT, this doesn't work (it does NOT satisfy $\vec{x}\,' = A\vec{x}$)

it turns out the correct form is, if $\vec{x_1} = e^{\lambda t}\vec{v_1}$

$$\boxed{\vec{x_2} = e^{\lambda t}(t\vec{v_1} + \vec{v_2})}$$

generalized eigenvector

ordinary eigenvector

now find $\vec{v}_2$

$\vec{x}\,' = A\vec{x}$ $\qquad \vec{x}_2 = e^{\lambda t}(t\vec{v}_1 + \vec{v}_2) = te^{\lambda t}\vec{v}_1 + e^{\lambda t}\vec{v}_2$

$\vec{x}_2$ must satisfy $\vec{x}\,' = A\vec{x}$

$\vec{x}_2\,' = t\lambda e^{\lambda t}\vec{v}_1 + e^{\lambda t}\vec{v}_1 + \lambda e^{\lambda t}\vec{v}_2$

$$t\lambda e^{\lambda t}\vec{v}_1 + e^{\lambda t}\vec{v}_1 + \lambda e^{\lambda t}\vec{v}_2 = Ate^{\lambda t}\vec{v}_1 + Ae^{\lambda t}\vec{v}_2$$

equate like terms

$te^{\lambda t}$ terms: $\lambda\vec{v}_1 = A\vec{v}_1$ $\rightarrow$ just definition of eigenvector
($\vec{v}_1$ is an eigenvector)

$e^{\lambda t}$ terms: $\vec{v}_1 + \lambda\vec{v}_2 = A\vec{v}_2$

$$\boxed{(A - \lambda I)\vec{v}_2 = \vec{v}_1}$$

if $\vec{v}_2$ were an eigenvector, then
right side is $\vec{0}$

so it's like an eigenvector but not
$\rightarrow$ generalized eigenvector

notice from $(A-\lambda I)\vec{v}_2 = \vec{v}_1$

multiply by $(A-\lambda I)$: $(A-\lambda I)^2\vec{v}_2 = (A-\lambda I)\vec{v}_1 = \vec{0}$

$$\boxed{(A-\lambda I)^2\vec{v}_2 = \vec{0}}$$

let's return to the example: $\vec{x}\,' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\vec{x}$

$\lambda = 1, 1 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

find $\vec{v}_2$: $(A-\lambda I)\vec{v}_2 = \vec{v}_1$

$$\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \to \vec{v}_2 = \begin{bmatrix} a \\ 1 \end{bmatrix}$$

choose ANY a: $a = 0$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

first solution: $e^{\lambda t}\vec{v}_1 = e^{t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

2nd solution: $e^{\lambda t}(t\vec{v}_1 + \vec{v}_2) = e^{t}\left(t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

gen. solution: $\boxed{c_1 e^{t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{t}\left(t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)}$

$$\boxed{\begin{aligned}(A-\lambda I)\vec{v_2} &= \vec{v_1}\\ (A-\lambda I)^2\vec{v_2} &= \vec{0}\end{aligned}}$$

2nd way to find $\vec{v_2}$: use the second equation

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\lambda = 1, 1$

$A-\lambda I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $(A-\lambda I)^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$(A-\lambda I)^2\vec{v_2} = \vec{0} \longrightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\vec{v_2}$ is arbitrary (almost)

pick ANY $\vec{v_2}$ EXCEPT $k\vec{v_1}$ or $\vec{0}$

let's choose $\vec{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

BUT, now we need to find $\vec{v_1}$ from $(A-\lambda I)\vec{v_2} = \vec{v_1}$

it may or may not be the ordinary eigenvector we found

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{v_1}$ (here, it matched the real eigenvector, but NOT always) solutions: $e^{\lambda t}\vec{v}_1 = e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$e^{\lambda t}(t\vec{v}_1 + \vec{v}_2) = e^t \left(t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

two ways summarized

$\vec{x}' = A\vec{x}$ $\quad \lambda = \lambda_1 \quad$ only one eigenvector $\vec{v}_1$

1. use the $\vec{v}_1$ we found to find $\vec{v}_2$

$$(A - \lambda I)\vec{v}_2 = \vec{v}_1$$

2. arbitrary choose $\vec{v}_2$ from $\underbrace{(A-\lambda I)^2}_{\text{zero matrix}} \vec{v}_2 = \vec{0}$

then find new $\vec{v}_1$ from $(A - \lambda I)\vec{v}_2 = \vec{v}_1$

for 3x3 and beyond, most of the process is the same

example $\vec{x}' = \begin{bmatrix} 2 & 0 & 0 \\ -4 & 6 & 4 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}$

$\lambda = \underbrace{2,\ 2,}_{\text{2 is repeated}}\ 6$

$\underline{\lambda = 6}$ $\quad (A - \lambda I)\vec{v} = \vec{0} \rightarrow \cdots \rightarrow \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Solution for $\lambda = 6$ (not repeated) is formed the usual way $\quad e^{6t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\underline{\lambda = 2}$ $\quad (A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -4 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

two free variables

$\rightarrow$ two eigenvectors

good!

$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Solution:

$$\vec{x}(t) = c_1 e^{6t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

<u>example</u>

$$\vec{x}\,' = \begin{bmatrix} -11 & 0 & -4 \\ -1 & -9 & -1 \\ 1 & 0 & -7 \end{bmatrix}\vec{x} \qquad \det(A-\lambda I) = 0$$

$$\lambda = -9, -9, -9$$

$$(A-\lambda I)\vec{v} = \vec{0}$$

$$\begin{bmatrix} -2 & 0 & -4 & 0 \\ -1 & 0 & -1 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

one free variable → one eigenvector

$$\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$ need 3, only have 1

let $\vec{v_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, need $\vec{v_2}$, $\vec{v_3}$

$$\boxed{\begin{aligned} (A-\lambda I)\vec{v_2} &= \vec{v_1} \\ (A-\lambda I)\vec{v_3} &= \vec{v_2} \end{aligned}}$$

find $\vec{v_2}$: $\left[\begin{array}{c|c} \underbrace{\begin{matrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{matrix}}_{(A-\lambda I)} & \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \end{array}\right]$

$\vec{v_1}$ (not $\vec{0}$ because $\vec{v_2}$ is not a true eigenvector)

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ -1 & 0 & -1 & 1 \\ -2 & 0 & -4 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \vec{v_2} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

free

(to be continued...)