7.6 Repeated Eigenvalues (part 2)

$\vec{x}\,' = A\vec{x}$ if A has repeated eigenvalues but has enough eigenvectors then solutions are formed normally : $e^{\lambda t}\vec{v}$

for example, $\vec{x}\,' = \begin{bmatrix} 2 & 0 & 0 \\ -4 & 6 & 4 \\ 0 & 0 & 2 \end{bmatrix}\vec{x}$

$\lambda = \underbrace{2, 2}_{\text{repeated}}, 6$ $\qquad \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ for $\lambda = 6$

for $\lambda = 2$, $(A - \lambda I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ -4 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

two free variables → two eigenvectors

$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Solutions: $e^{6t}\begin{bmatrix}0\\1\\0\end{bmatrix}$, $e^{2t}\begin{bmatrix}1\\1\\0\end{bmatrix}$, $e^{2t}\begin{bmatrix}1\\0\\1\end{bmatrix}$

if there are not enough eigenvectors, then we need to supplement with generalized eigenvectors.

<u>example</u>

$$\vec{x}' = \begin{bmatrix} -11 & 0 & -4 \\ -1 & -9 & -1 \\ 1 & 0 & -7 \end{bmatrix} \vec{x}$$

$$\lambda = -9, -9, -9$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

$$\begin{bmatrix} -2 & 0 & -4 & 0 \\ -1 & 0 & -1 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

$$\to \cdots \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2 pivots

1 free variable

↳ 1 eigenvector

missing <u>Two</u>

the only ordinary eigenvector is $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

missing $\vec{v}_2$, $\vec{v}_3$ → generalized eigenvectors

$(A-\lambda I)\vec{v}_1 = \vec{0}$ because $\vec{v}_1$ is an ordinary eigenvector

$(A-\lambda I)\vec{v}_2 = \vec{v}_1$

$(A-\lambda I)\vec{v}_3 = \vec{v}_2$

$$A-\lambda I = \begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\begin{array}{c} \begin{array}{cccc} & & & \vec{v}_1 \end{array} \\ \left[\begin{array}{ccc|c} -2 & 0 & -4 & 0 \\ -1 & 0 & -1 & 1 \\ 1 & 0 & 2 & 0 \end{array}\right] \end{array} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\vec{v}_2 = \begin{bmatrix} -2 \\ r \\ 1 \end{bmatrix}$ choose ANY r

let's choose 0

$\vec{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$(A-\lambda I)\vec{v}_3 = \vec{v}_2$

$$\begin{array}{c} \begin{array}{cccc} & & & \vec{v}_2 \end{array} \\ \left[\begin{array}{ccc|c} -2 & 0 & -4 & -2 \\ -1 & 0 & -1 & 0 \\ 1 & 0 & 2 & 1 \end{array}\right] \end{array} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ 2nd element is free, chose 0

now we form the three solutions

1st : $e^{\lambda t}\vec{v}_1 = e^{-9t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

2nd : $e^{\lambda t}(t\vec{v}_1 + \vec{v}_2) = e^{-9t}\left(t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right)$

3rd : $e^{\lambda t}(\frac{1}{2}t^2\vec{v}_1 + t\vec{v}_2 + \vec{v}_3) = e^{-9t}\left(\frac{t^2}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$

gen. solution is linear combo of them.

the second way to find these generalized eigenvectors:

$$(A-\lambda I)\vec{v}_1 = \vec{0} \qquad \vec{v}_1 \text{ is true eigenvector}$$

$$(A-\lambda I)\vec{v}_2 = \vec{v}_1 \qquad \vec{v}_2, \vec{v}_3 : \text{generalized eigenvectors}$$

$$(A-\lambda I)\vec{v}_3 = \vec{v}_2$$

multiply by $(A-\lambda I)$

$$(A-\lambda I)^2\vec{v}_3 = (A-\lambda I)\vec{v}_2 = \vec{v}_1$$

multiply by $(A-\lambda I)$ again

$$(A-\lambda I)^3\vec{v}_3 = (A-\lambda I)\vec{v}_1 = \vec{0}$$

$$A-\lambda I = \begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$(A-\lambda I)^3 = \begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} -2 & 0 & 4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} \text{same} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$(A-\lambda I)^{k+1}$ is ALWAYS a matrix of zeros if there are k eigenvectors missing for this λ

$$\overbrace{(A-\lambda I)^3}^{\text{all zeros}} \vec{v}_3 = \vec{0}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad a, b, c \text{ are free}$$

choose ANY vector to be $\vec{v}_3$
EXCEPT zero vector or any
multiple of the true eigenvector

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

so, let's use $\boxed{\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$

now find $\vec{v}_2$ from $(A-\lambda I)\vec{v}_3 = \vec{v}_2$

$$\begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \qquad \boxed{\vec{v}_2 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}}$$

now, find $\vec{v}_1$ from $(A-\lambda I)\vec{v}_2 = \vec{v}_1$ even though we have already found a true eigenvector

$$\begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

this happens to be the same as the true eigenvector we found, but this is NOT always the case.

$$\boxed{\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

solutions: $e^{\lambda t}\vec{v}_1 = e^{-8t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$e^{\lambda t}(t\vec{v}_1 + \vec{v}_2) = e^{-8t}\left(t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}\right)$$

$$e^{\lambda t}\left(\frac{t^2}{2}\vec{v}_1 + t\vec{v}_2 + \vec{v}_3\right) = e^{-8t}\left(\frac{t^2}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

Solutions will look different depending on choice of $\vec{v}_3$ but the general solution is the same (c_1, c_2, c_3 will adjust for the different forms)

Summary of two methods

first, find the true eigenvector

method 1 : use true eigenvector as $\vec{v_1}$, then

find the rest from $(A-\lambda I)\vec{v_2} = \vec{v_1}$

$(A-\lambda I)\vec{v_3} = \vec{v_2}$

method 2 : $(A-\lambda I)^{k+1}$ = zero matrix (k eigenvectors missing)

choose the top one to be whatever EXCEPT zero vector or any multiple of true eigenvector

then $(A-\lambda I)\vec{v_3} = \vec{v_2}$

$(A-\lambda I)\vec{v_2} = \vec{v_1}$ ← use this as $\vec{v_1}$ even if it is not the true eigenvector