7.6 Repeated Eigenvalues (continued)

from last time: $\vec{x}' = \begin{bmatrix} -11 & 0 & -4 \\ -1 & -9 & -1 \\ 1 & 0 & -7 \end{bmatrix} \vec{x}$

$\lambda = -9, -9, -9$

one ordinary eigenvector: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

missing two eigenvectors

one way to find the missing ones:

let $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (ordinary eigenvector)

the other two are generalized ones

$(A - \lambda I)\vec{v}_2 = \vec{v}_1$

$(A - \lambda I)\vec{v}_3 = \vec{v}_2$

$$\left[\begin{array}{ccc|c} -2 & 0 & -4 & 0 \\ -1 & 0 & -1 & 1 \\ 1 & 0 & 2 & 0 \end{array}\right]$$

$$\to \cdots \to \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \therefore \vec{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \leftarrow \text{free}$$

now $(A - \lambda I)\vec{v}_3 = \vec{v}_2$

$$\begin{bmatrix} -2 & 0 & -4 & -2 \\ -1 & 0 & -1 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix} \to \cdots \to \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \leftarrow \text{free}$$

Solutions: $\vec{x}_1 = e^{\lambda t}\vec{v}_1 = e^{-5t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\vec{v}_1$ is ordinary eigenvector

$$\vec{x}_2 = e^{\lambda t}(t\vec{v}_1 + \underbrace{\vec{v}_2}_{\text{generalized eigenvector}}) = e^{-5t}\left(t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$\vec{x}_3 = e^{\lambda t}\left(\frac{t^2}{2}\vec{v}_1 + t\vec{v}_2 + \vec{v}_3\right)$$

$$= e^{-5t}\left(\frac{t^2}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$$

general solution: $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$

2nd method: find $\vec{v_3}$ first, then find $\vec{v_2}$, $\vec{v_1}$

$$(A-\lambda I)\vec{v_1} = \vec{0}$$ because $\vec{v_1}$ is an ordinary e-vector

$$(A-\lambda I)\vec{v_2} = \vec{v_1}$$

$$(A-\lambda I)\vec{v_3} = \vec{v_2}$$

multiply by $(A-\lambda I)$

$$(A-\lambda I)^2\vec{v_3} = (A-\lambda I)\vec{v_2} = \vec{v_1}$$

multiply again

$$(A-\lambda I)^3\vec{v_3} = (A-\lambda I)\vec{v_1} = \vec{0} \rightarrow (A-\lambda I)^3\vec{v_3} = \vec{0}$$

$$(A-\lambda I)^3 = \begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

if A is missing n e-vectors then $(A-\lambda I)^{n+1} = [0]$ zero matrix

$$(A-\lambda I)^3\vec{v_3} = \vec{0} \qquad \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underbrace{\begin{matrix} 0 & 0 & 0 \end{matrix}}_{(A-\lambda I)^3} \quad \underbrace{0}_{\vec{0}}$$

$\rightarrow \vec{v_3}$ is arbitrary (all components free) pick ANY $\vec{v_3}$ EXCEPT any linear combo or multiple of ordinary eigenvector or $\vec{0}$

ord. e-vector : $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ pick $\boxed{\vec{v_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$ $\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ also ok} \right)$

now $\vec{v_2}$: $(A-\lambda I)\vec{v_3} = \vec{v_2}$

$$\begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \vec{v_2}}$$

now $\vec{v_1}$: $(A-\lambda I)\vec{v_2} = \vec{v_1}$

do this even though we have ord. e-vector already : the $\vec{v_1}$ here may or may not be the ord. e-vector → use this for solution

$$\begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

matches the ord. one but this doesn't always happen

solutions: $\vec{x_1} = e^{\lambda t}\vec{v_1}$

$\vec{x_2} = e^{\lambda t}(t\vec{v_1} + \vec{v_2})$

$\vec{x_3} = e^{\lambda t}\left(\frac{t^2}{2}\vec{v_1} + t\vec{v_2} + \vec{v_3}\right)$

now : A is 3x3, missing one e-vector

$$\vec{x}\,' = \begin{bmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{bmatrix} \vec{x}$$

$\lambda = 1, 1, 1$

two ordinary eigenvectors: $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$

ordinary e-vectors

first two solutions are easy: $e^{\lambda t}\vec{v}_1 = e^{t}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$e^{\lambda t}\vec{v}_2 = e^{t}\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$

the missing solution looks like this:

$$\boxed{e^{\lambda t}\left(t\vec{u} + \vec{v}_3\right)}$$

some linear combo of the ordinary ones

$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2$ and $\boxed{(A-\lambda I)\vec{v}_3 = \vec{u}}$

First method: find $\vec{u}$ from $\vec{v}_1, \vec{v}_2$ (ordinary e-vectors)

$$(A-\lambda I) = \begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

$$(A-\lambda I)\vec{v}_3 = \vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 = \begin{bmatrix} c_1 \\ 2c_2 \\ 2c_1 - 3c_2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & -3 & -2 & c_1 \\ 8 & -6 & -4 & 2c_2 \\ -4 & 3 & 2 & 2c_1 - 3c_2 \end{array}\right]$$

$\rightarrow$ consistent only if 2nd row is $2 \times$ 1st

$\rightarrow 2c_2 = 2 \cdot c_1 \rightarrow c_1 = c_2$ don't want $\vec{u} = \vec{0}$

now choose c_1 or c_2

here, $c_1 = 2$ so $c_2 = 2$

$$\left[\begin{array}{ccc|c} 4 & -3 & -2 & 2 \\ 8 & -6 & -4 & 4 \\ -4 & 3 & 2 & -2 \end{array}\right] \Rightarrow (A-\lambda I)\vec{v}_3 = \vec{u}$$

$\vec{u}$

$$\to \cdots \to \begin{bmatrix} 1 & -3/4 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} \frac{1}{2} + \frac{3}{4}a + \frac{1}{2}b \\ a \\ b \end{bmatrix}$$

choose $a=0,\ b=-1$

$$\boxed{\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}}$$

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2$$

$$= 2\cdot\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 2\cdot\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

3rd solution: $e^{\lambda t}(t\vec{u} + \vec{v}_3)$

$$= \boxed{e^{t}\left(t\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}\right)}$$

1st & 2nd: $e^{\lambda t}\vec{v}_1,\ e^{\lambda t}\vec{v}_2$

2nd method : find $\vec{v}_3$, then find $\vec{u}$ from $(A-\lambda I)\vec{v}_3 = \vec{u}$

same knowns:

$$A-\lambda I = \begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix} \qquad \lambda = 1, 1, 1$$

ordinary eigenvectors : $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$

$$(A-\lambda I)\vec{v}_1 = \vec{0}$$

$$(A-\lambda I)\vec{v}_2 = \vec{0}$$

$$(A-\lambda I)\vec{v}_3 = \vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2$$

multiply by $A-\lambda I$

$$(A-\lambda I)^2\vec{v}_3 = c_1 \underbrace{(A-\lambda I)\vec{v}_1}_{\vec{0}} + c_2 \underbrace{(A-\lambda I)\vec{v}_2}_{\vec{0}} = \vec{0}$$

$$(A-\lambda I)^2\vec{v}_3 = \vec{0}$$

$$(A-\lambda I)^{2} = \begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix}$$

\# of missing e-vector +1

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A-\lambda I)^2 \vec{v}_3 = \vec{0} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\vec{v}_3$ is arbitrary except: NOT $\vec{0}$

NOT dep. from either true e-vectors

so, pick something linearly indp. from $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ +2 \\ -3 \end{bmatrix}$$

$$\boxed{\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \text{ or } \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

now find $\vec{u}$: $(A-\lambda I)\vec{v}_3 = \vec{u}$

$$\begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}$$

finally. $$\boxed{\underbrace{\vec{x}_3 = e^{\lambda t}(t\vec{u} + \vec{v}_3)}_{\text{3rd solution}} = e^t\left(t\begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)}$$

first two: $$e^{\lambda t}\vec{v}_1 = e^t\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$e^{\lambda t}\vec{v}_2 = e^t\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$