7.6 Repeated Eigenvalues (part 3)

last time: λ repeated 3 times, missing 2 eigenvectors

today: λ repeated 3 times, missing 1 eigenvector

$$\vec{x}' = \begin{bmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{bmatrix} \vec{x}$$

$\lambda = 1, 1, 1$

ordinary eigenvectors

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

missing one

Two methods

1. using ordinary eigenvector and use $(A-\lambda I)\vec{v}_2 = \vec{v}_1$ etc to build other solutions "up"
2. choose the "top" one arbitrarily and use $(A-\lambda I)\vec{v}_3 = \vec{v}_2$ etc to build others "down"

1st method

two solutions: $\vec{x}_1 = e^{\lambda t}\vec{v}_1 = e^{t}\begin{bmatrix}1\\0\\2\end{bmatrix}$

$\vec{x}_2 = e^{\lambda t}\vec{v}_2 = e^{t}\begin{bmatrix}0\\2\\-3\end{bmatrix}$

missing $\vec{x}_3$

third one is: $\vec{x}_3 = e^{\lambda t}(t\vec{u} + \vec{v}_3)$

some linear combo of the two ordinary eigenvectors

$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2$

and $(A-\lambda I)\vec{v}_3 = \vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2$

first, find what c_1 and c_2 are:

$(A-\lambda I)\vec{v}_3 = \vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2$

$$c_1\begin{bmatrix}1\\0\\2\end{bmatrix} + c_2\begin{bmatrix}0\\2\\-3\end{bmatrix} = \begin{bmatrix}c_1\\2c_2\\2c_1-3c_2\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}4 & -3 & -2 & c_1\\ 8 & -6 & -4 & 2c_2\\ -4 & 3 & 2 & 2c_1-3c_2\end{array}\right]$$

look at row 1 and row 2

system is consistent only if $2c_1 = 2c_2 \rightarrow c_1 = c_2$

choose any c_1, c_2 such that $c_1 = c_2$

let's choose $c_1 = 2$ for convenience ($c_1 = c_2$ so $c_2 = 2$)

$(A - \lambda I)\vec{v}_3 = \vec{u}$ becomes

$$\left[\begin{array}{ccc|c} 4 & -3 & -2 & 2 \\ 8 & -6 & -4 & 4 \\ -4 & 3 & 2 & -2 \end{array}\right] \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & -3/4 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\vec{v} = \cancel{\begin{bmatrix} a \\ b \\ c \end{bmatrix}} = \begin{bmatrix} \frac{1}{2} + \frac{3}{4}a + \frac{1}{2}b \\ a \\ b \end{bmatrix}$$

2nd, 3rd elements are free

choose $a = 0$, $b = -1$

$$\boxed{\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}}$$

$$\vec{x}_3 = e^{\lambda t}(t\vec{u} + \vec{v}_3)$$

$$\downarrow$$

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

$$\vec{x}_3 = e^t\left(t\begin{bmatrix}2\\4\\-2\end{bmatrix}+\begin{bmatrix}0\\0\\-1\end{bmatrix}\right)$$

general solution:

$$\boxed{\vec{x} = c_1 e^t\begin{bmatrix}1\\0\\2\end{bmatrix} + c_2 e^t\begin{bmatrix}0\\2\\-3\end{bmatrix} + c_3 e^t\left(t\begin{bmatrix}2\\4\\-2\end{bmatrix}+\begin{bmatrix}0\\0\\-1\end{bmatrix}\right)}$$

<u>2nd method:</u> $\vec{v}_1, \vec{v}_2$ are ordinary eigenvectors

$$\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\2\\-3\end{bmatrix}$$

but we don't necessarily use them in our solutions

$$(A-\lambda I)\vec{v}_3 = \vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2$$

multiply by $(A-\lambda I)$ on both sides

$$(A-\lambda I)^2\vec{v}_3 = c_1\underbrace{(A-\lambda I)\vec{v}_1}_{\vec{0}} + c_2\underbrace{(A-\lambda I)\vec{v}_2}_{\vec{0}}$$

$$\boxed{(A-\lambda I)^2\vec{v}_3 = \vec{0}}$$

$$A-\lambda I = \begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix}$$

$$(A-\lambda I)^2 = \begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix}\begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A-\lambda I)^2\vec{v}_3 = \vec{0} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\vec{v}_3$ is arbitrary*

let's choose $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

but $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ is ok, too

(and many other choices)

*: not $\vec{0}$ and must be linearly indp from existing eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$

using this "top down" method, we don't necessarily reuse the ordinary eigenvectors, we MUST find them this way:

$$(A-\lambda I)\vec{v}_3 = \vec{v}_2$$

$$\begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} = \vec{v}_2$$

usually we then do $(A-\lambda I)\vec{v}_2 = \vec{v}_1$

but here, $\begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

because $\vec{v}_2$ is a linear combo of the two ordinary eigenvectors

do NOT use $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ to be $\vec{v}_1$, choose either of the two ordinary eigenvectors to be $\vec{v}_1$

here, let's choose $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

solutions: $\vec{x}_1 = e^{\lambda t}\vec{v}_1 = e^{t}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

the one we found on last page

$\vec{x}_2 = e^{\lambda t}\vec{v}_2 = e^{t}\begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}$

$\vec{x}_3 = e^{\lambda t}(\vec{v}_2 t + \vec{v}_3) = e^{\cancel{\lambda} t}\left(t\begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$

general solution:

$$\boxed{\vec{X} = c_1 e^{t}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} + c_3 e^{t}\left(t\begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)}$$