1.4 Separable Differential Eqs

revisit $\frac{dy}{dx} = 2x$ solution is easy: $y = \int 2x\,dx = x^2 + c$

because we can isolate x on its own side

a separable eq. is one that we can do something to isolate x from y. Then we integrate both sides.

for example: $\frac{dy}{dx} = xy$

think of $\frac{dy}{dx}$ as a fraction (technically not true but the end result is the same)

$$dy = xy\,dx$$

$\frac{1}{y}\,dy = x\,dx$ now integrate both sides

$$\int \frac{1}{y}\,dy = \int x\,dx$$

$$\ln|y| + C_1 = \frac{1}{2}x^2 + C_2$$

goal: solve for y as a function of x

(explicit solution)

or an equation relating

x and y (implicit solution)

$$\ln|y| = \frac{1}{2}x^2 + \boxed{C_2 - C_1}$$

still a constant

call it C

$$\ln|y| = \frac{1}{2}x^2 + C$$

$$e^{\ln|y|} = e^{\frac{1}{2}x^2 + C}$$

$$|y| = e^{\frac{1}{2}x^2} \cdot \boxed{e^C}$$

still a constant

call it C (we abuse the letter C)

$$|y| = Ce^{\frac{1}{2}x^2}$$

$$\boxed{y = Ce^{\frac{1}{2}x^2}}$$

explicit form

example

$$(1+x)y' = y$$

$$(1+x)\frac{dy}{dx} = y$$

use multiplication or division to separate x and y

$$\frac{1}{y}\,dy = \frac{1}{x+1}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{1+x}\,dx$$

$$\ln|y| = \ln|1+x| + C$$

$$e^{\ln|y|} = e^{\ln|1+x|}\cdot e^{C}$$

$$|y| = C\,|1+x|$$

$$\boxed{y = C(1+x)}$$

C depends on initial condition

not all differential eqs. are separable

for example, $\frac{dy}{dx} = x + y$ can't separate x and y by multiplication or division

never do addition / subtraction

$$\frac{dy}{dx} = x + y$$

$$\frac{dy}{dx} - y = x$$

$$dy - \underbrace{y\,dx}_{?} = x\,dx$$

<u>example</u>

$$\frac{dy}{dx} = xy^2 - y^2 \qquad y(1)=2$$

$$= y^2(x-1)$$

$$\frac{1}{y^2}\,dy = (x-1)\,dx$$

$$\int \frac{1}{y^2}\,dy = \int (x-1)\,dx$$

$$-\frac{1}{y} = \frac{1}{2}x^2 - x + C$$

$$y = \frac{-1}{\frac{1}{2}x^2 - x + C} \qquad \text{general}$$

$$y(1)=2 \rightarrow x=1 \quad y=2$$

$$2 = \frac{-1}{\frac{1}{2} - 1 + C} = \frac{-1}{-\frac{1}{2}+C} \qquad -\frac{1}{2}+C = -\frac{1}{2}$$

so $C=0$

$$\boxed{y = \frac{-1}{\frac{1}{2}x^2 - x}} \qquad \text{particular}$$

applications: exponential growth / decay

$\frac{dy}{dt} = ky$ grows/decays at a rate proportional to its size (population, interest radioactive decay)

k: constant of proportionality

$k > 0 \to$ growth $\left(\frac{dy}{dt} > 0\right)$

$k < 0 \to$ decay $\left(\frac{dy}{dt} < 0\right)$

for example, for Uranium 235

$k = -3.12 \times 10^{-17}$

Newton's Law of Cooling

$T(t)$: temp. as function of t

M : temp. of surrounding medium (constant)

$$\frac{dT}{dt} = k(\underbrace{M}_{\text{surrounding temp.}} - T)$$

as long as $M \neq T$, $\frac{dT}{dt} \neq 0$ until $M = T$

over time, temp. of object → temp. of surrounding

k : insulation or lack of

this is separable! Let's solve it.

k, M are constants

$$\int \frac{1}{M-T}\, dT = \int k\, dt$$

$$-\ln|M-T| = kt + C \qquad \text{try to solve for } T$$

$$\ln|M-T| = -kt + C$$

$$M - T = e^{-kt} \cdot e^{C}$$

$$M - T = C e^{-kt}$$

$$\boxed{T = M - Ce^{-kt}}$$

$(k > 0)$

$$\lim_{t\to\infty} e^{-kt} = 0$$

$$\lim_{t\to\infty} T = M$$

another application: terminal ~~termi~~ velocity

$$\frac{dv}{dt} = g - \frac{c}{m}v \qquad \text{also separable}$$

$$\frac{1}{g - \frac{c}{m}v}\, dv = dt$$