1.5 First-order Linear Eq.

first order linear: $\dfrac{dy}{dx} + \underline{\underline{P(x)}}\, y = \underline{\underline{Q(x)}}$

cannot contain y

so, $y' + xy = 1$ is linear

but $y' + y^2 = 2$ is NOT

$y' + \underline{\underline{y}}\,y = 2$

contains y

Solution method: ~~alwau~~ we can always find an <u>integrating factor</u> such that when multiplied to both sides the left can always be turned into the product of something. Then integrate both sides to solve for y.

example

$$y' + \boxed{\frac{1}{x}}\, y = \boxed{0}$$

no y in these

notice if we multiply by $\boxed{x}$ integrating factor

$$\underbrace{x y' + y} = 0$$

$$\frac{d}{dx}(xy) = 0 \quad \text{by product rule}$$

integrate both sides

$$xy = C$$

so, $\boxed{y = \frac{C}{x}}$

How to find the integrating factor ?

eg : $y' + Py = Q$ $\quad$ P, Q do NOT contain y

want to multiply I such that the left side turns into

$$\frac{d}{dx}(Iy) = \underbrace{Iy' + I'y}_{\text{we want the left side to be this}} \quad \text{product rule}$$

$$I(y' + Py) = IQ$$

$$\underbrace{Iy' + IPy}_{\text{equal}} = IQ$$

so, we want : $Iy' + I'y = Iy' + IPy$

$$I' = IP \rightarrow \text{solve this for } I$$

$$I' = IP$$

$$\frac{dI}{dx} = I\,P(x)$$ separable in I and x

$$\frac{1}{I}\,dI = P\,dx$$

$$\int \frac{1}{I}\,dI = \int P\,dx$$

$$\ln|I| = \int P\,dx$$

So, $\boxed{I = e^{\int P\,dx}}$ integrating factor

to turn left side into $\frac{d}{dx}(Iy)$

example ~~$(1+x)y' =$~~

$$xy' + 3y = x \qquad y(1) = 2$$

not 1, so divide by this to put into standard form

$$y' + Py = Q$$

$$y' + \frac{3}{x}y = 1 \qquad \text{so, } P = \frac{3}{x}$$

ALWAYS identify P in standard form

$$I = e^{\int P\,dx} = e^{\int \frac{3}{x}\,dx}$$

$$= e^{3\ln x}$$

+C is not needed here (but including it does not hurt)

$$= e^{\ln x^3} = \boxed{x^3}$$

multiply both sides by $I = x^3$

$$x^3\left(y' + \frac{3}{x}y\right) = x^3 \cdot 1$$

$$\underbrace{x^3y' + 3x^2y}_{\frac{d}{dx}(x^3y)} = x^3$$

if I is correct, left side is ALWAYS $\frac{d}{dx}(Iy)$

so, $\frac{d}{dx}(x^3y) = x^3$

$$x^3y = \int x^3\,dx$$

$$= \frac{1}{4}x^4 + C$$

$$y = \frac{1}{4}x + \frac{C}{x^3}$$ general

use $y(1)=2$ to find C : $2 = \frac{1}{4} + C$ so, $C = \frac{7}{4}$

$$\boxed{y = \frac{1}{4}x + \frac{7}{4}\cdot\frac{1}{x^3}}$$

why is +C not needed when finding $I = e^{\int p\,dx}$?

$$y' + \frac{3}{x}y = 1$$

$$\begin{aligned} I = e^{\int p\,dx} &= e^{\int \frac{3}{x}dx} \\ &= e^{3\ln x + C} \\ &= e^{\ln x^3} \cdot e^{C} \\ &= x^3 \cdot A \end{aligned}$$

1 if no +C above

multiply both sides by $I = Ax^3$

$$\cancel{A}x^3\left(y' + \frac{3}{x}y\right) = \cancel{A}x^3 \cdot 1$$

cancel out, as if $A = 1$ (which means $C = 0$ in I)

example $y' - 4y = e^x$

↑ coefficient is 1 so is in standard form

$P(x) = -4$ $\qquad I = e^{\int P\,dx} = e^{\int -4\,dx} = e^{-4x}$

(no +C or use C=0)

multiply I : $e^{-4x}(y' - 4y) = e^{-4x} \cdot e^x$

$$\underbrace{e^{-4x}y' - 4e^{-4x}y}_{\frac{d}{dx}(e^{-4x}y)} = e^{-3x}$$

$$\frac{d}{dx}(e^{-4x}y) = e^{-3x}$$

integrate: $e^{-4x}y = \int e^{-3x}\,dx$

$$= -\frac{1}{3}e^{-3x} + C$$

this +C is important

$$\boxed{y = -\frac{1}{3}e^{x} + Ce^{4x}}$$

terminal velocity : $\frac{dv}{dt} = g - \frac{c}{m}v$

this is linear: $v' + \underbrace{\boxed{\frac{c}{m}}}_{P} v = \underbrace{\boxed{g}}_{Q}$ neither contains V

$$I = e^{\int \frac{c}{m} dt} = e^{\frac{c}{m}t}$$

$$\underbrace{e^{\frac{c}{m}t}\left(v' + \frac{c}{m}v\right)} = g \cdot e^{\frac{c}{m}t}$$

$$\frac{d}{dx}\left(e^{\frac{c}{m}t} v\right) = g \cdot e^{\frac{c}{m}t}$$

$$e^{\frac{c}{m}t} v = \frac{m}{c} g e^{\frac{c}{m}t} + c_1$$

$$V = \frac{mg}{c} + c_1 e^{-\frac{c}{m}t}$$