1.5 First-order Linear Eq. (continued)

$y' + py = Q$

integrating factor $I = e^{\int p\,dx}$

$\frac{d}{dx}(Iy) = IQ$ integrate to solve for y.

application : mixing problem

example : A tank initially contains 40 lbs of salt dissolved in 600 gals of water.

Water containing $\frac{1}{2}$ lb of salt per gallon is poured in at the rate of 4 gal/min.

The well-mixed solution is let out of the tank at 4 gal/min.

Find an eq. describing the amount of salt in the tank.

let $y(t)$ be the amount of salt in lb as function of time

problem tells us the rates of change

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

rate of change of salt

$$\frac{\text{amount of salt}}{\text{volume}} = \frac{y}{600}$$

4 in, 4 out

$$= \underbrace{\left(\tfrac{1}{2}\ \text{lb/gal}\right)\left(4\ \text{gal/min}\right)}_{\text{salt water going in}} - \underbrace{\left(\frac{y}{600}\ \text{lb/gal}\right)\left(4\ \text{gal/min}\right)}_{\text{salt coming out water}}$$

(concentration, flow rate; concentration, flow rate)

$$\boxed{\frac{dy}{dt} = 2 - \frac{y}{150}}$$

$$\boxed{y(0) = 40}$$

differential eq. for y

solve for y

rewrite: $y' + \underbrace{\frac{1}{150}}_{P} y = \underbrace{2}_{Q}$

$$I = e^{\int P\,dt} = e^{\int \frac{1}{150}\,dt} = e^{\frac{1}{150}t}$$

$$e^{\frac{1}{150}t} y' + \frac{1}{150} e^{\frac{1}{150}t} y = 2e^{\frac{1}{150}t}$$

$$\frac{d}{dt}\left(e^{\frac{1}{150}t} y\right) = 2e^{\frac{1}{150}t}$$

$$e^{\frac{1}{150}t} y = \int 2e^{\frac{1}{150}t}\,dt = 300e^{\frac{1}{150}t} + c$$

$$y = 300 + ce^{-\frac{1}{150}t} \qquad y(0) = 40$$

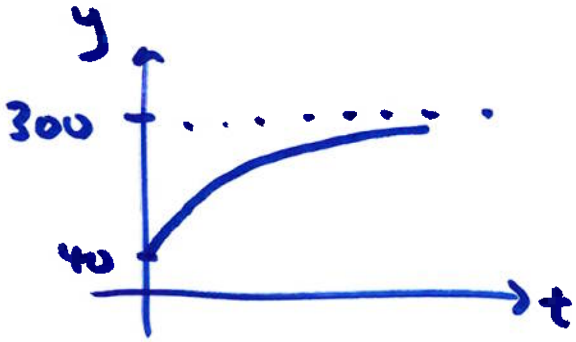
$$40 = 300 + C \qquad C = -260$$

$$\boxed{y = 300 - 260e^{-\frac{1}{150}t}}$$

$$\lim_{t\to\infty} y = 300 \text{ lb}$$

$$\frac{300 \text{ lb}}{600 \text{ gal}} = \frac{1}{2} \text{ lb/gal}$$

concentration of salt water in the tank approaches the concentration of the incoming flow



after a long time, the tank is essentially a pipe because same flow rate in and out.

now let's change the flow rate out to 6 gal/min

in: same $\rightarrow$ $\frac{1}{2}$ lb/gal, 4 gal/min

out: $\dfrac{\text{amount in tank}}{\text{volume in tank}}$ lb/gal, 6 gal/min

now volume changes: start w/ 600

4 in per min, 6 out per min

$V(t) = 600 - 2t$

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

$$= \left(\tfrac{1}{2}\right)(4) - \left(\frac{y}{600-2t}\right)(6)$$

$$y' = 2 - \frac{3}{300-t}\, y$$

$$\boxed{y' + \frac{3}{300-t}\, y = 2 \quad , \quad y(0) = 40}$$

Solve this for y

$$I = e^{\int p\,dt} = e^{\int \frac{3}{300-t}\,dt} = e^{-3\ln(300-t)} = e^{\ln(300-t)^{-3}}$$

$$= (300-t)^{-3} = \frac{1}{(300-t)^3}$$

$$\frac{1}{(300-t)^3}\, y' + \frac{3}{300-t} \cdot \frac{1}{(300-t)^3}\, y = \frac{2}{(300-t)^3}$$

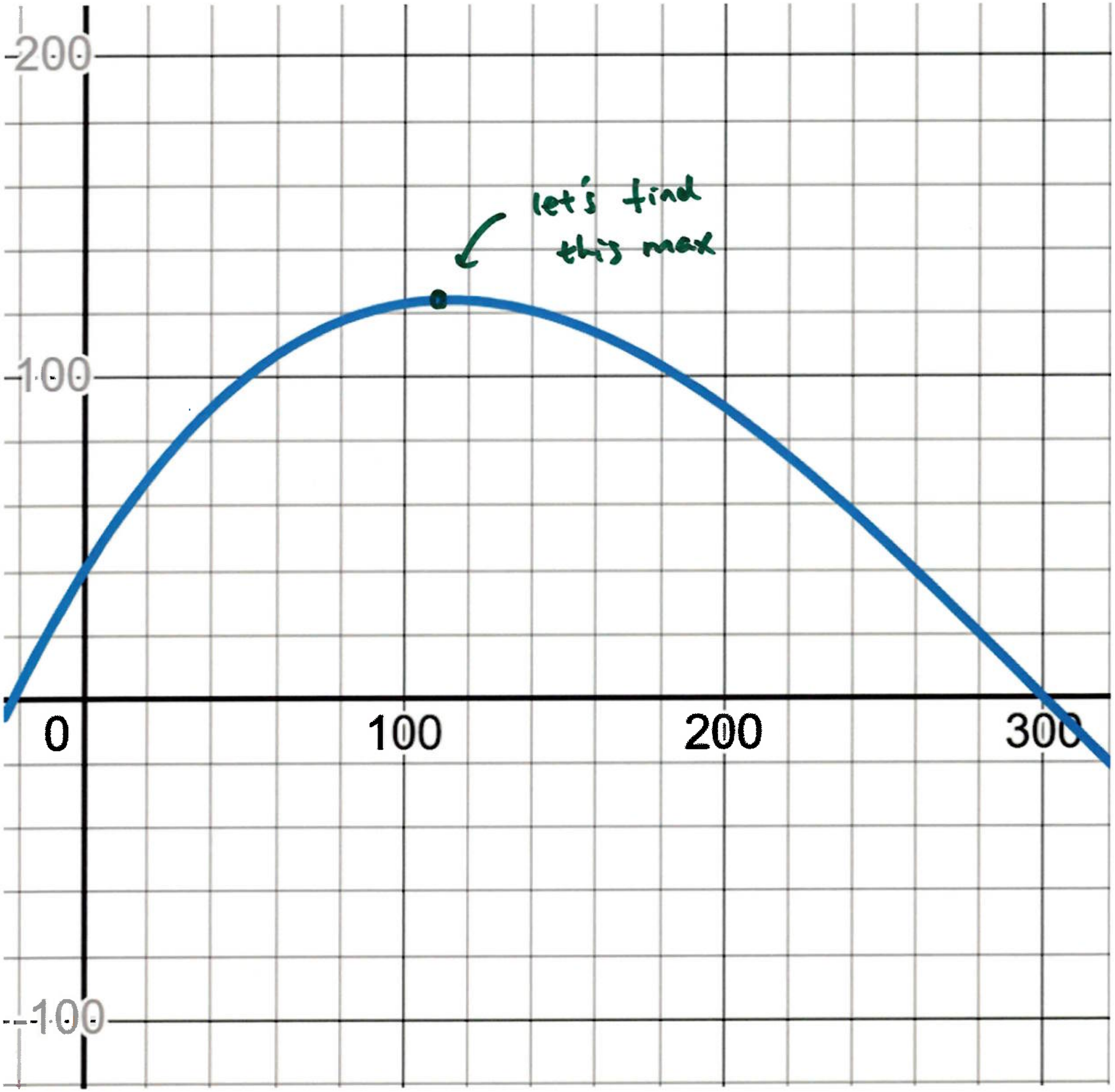
$$\frac{d}{dt}\left[\frac{1}{(300-t)^3}\,y\right] = \frac{2}{(300-t)^3}$$

$$\frac{1}{(300-t)^3}\,y = \frac{1}{(300-t)^2} + C$$

$$y = 300-t + C(300-t)^3 \qquad y(0)=40$$

$$40 = 300 + C(300)^3 \qquad C = -\frac{13}{1350000}$$

$$\boxed{y = 300-t - \frac{13}{1350000}(300-t)^3}$$


200
100
-100
0
100
200
300
let's find this max

$$y = 300 - t - \frac{13}{1350000}(300-t)^3$$

find t at which y is maximized, and the max y

$$y' = -1 + \frac{39}{1350000}(300-t)^2 = 0$$

solve for t: $t \approx 114$

max y is y at $t = 114$

$$y_{max} \approx 124$$

bring back the differential eq.

$$y' = 2 - \frac{3}{300-t}y \longrightarrow \text{max } y \text{ at } y' = 0$$

plug in y_{max}, $t \approx 114$ $y' = 0$

max reached when in matched by out

since $t \approx 114$ is when y is max

$t < 114$, $y' > 0$ → saltier water in than out

$t > 114$ $y' < 0$ → saltier water out

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what if flow rate out is 2 gal/min

flow in as before volume $= 600 + 2t$

diff. eq: $y' = 2 - \frac{y}{600+2t}(2)$ $y(0) = 40$

$\vdots$

$$y = \frac{t^2 + 600t + 12000}{t + 300}$$ $\lim_{t\to\infty} y = \infty$

this is a tank w/ no hole or a small hole
as hole gets larger salt accumulation slows
when hole is at the size where flow in = ~~rate~~ flow out → 1st example
when hole is larger → 2nd example