1.5 First-order Linear Differential Eq. (continued)

recap: $y' + P(x)y = Q(x)$

solve by multiplying by integrating factor

$$I = e^{\int P(x)dx} \quad \text{(no } +c\text{)}$$

then the differential eq. turns into

$$\frac{d}{dx}(Iy) = IQ$$

solve by integration.

Let's find out why $I = e^{\int P(x)dx}$

$$y' + Py = Q$$

multiply by I : $Iy' + IPy = IQ$

turn into : $\frac{d}{dx}(Iy) = IQ$

so, $Iy' + Ipy = \frac{d}{dx}(Iy)$

$= Iy' + I'y$

so, $Ipy = I'y$

or $I' = Ip$ separable

$$\frac{dI}{dx} = Ip$$

$$\frac{1}{I}\,dI = p\,dx$$

$$\int \frac{1}{I}\,dI = \int p\,dx$$

$$\ln I = \int p\,dx \rightarrow \boxed{I = e^{\int p\,dx}}$$

example: mixing problem

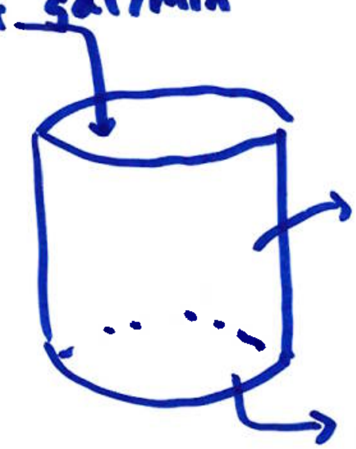
A tank initially contains 40 pounds of salt dissolved in 600 gallons of water.

Water containg 0.5 pounds of salt per gallon is poured in at 4 gallons/min.

The well-stirred solution is let out of the tank at 4 gallons/min.

Find: differential eq. describing amount of salt ~~of~~ in the tank

then solve the differential eq.



let $y(t)$ be amount of salt in the tank

rate of change of salt $\rightarrow$ $\frac{dy}{dt}$ = (rate in) − (rate out)

$$= \underbrace{(0.5\ \text{lb/gal})}_{\text{concentration}}\underbrace{(4\ \text{gal/min})}_{\text{flow rate}} - \underbrace{\left(\frac{y}{600}\ \text{lb/gal}\right)}_{\text{conc.}}\underbrace{(4\ \text{gal/min})}_{\text{flow rate}}$$

$$\boxed{\frac{dy}{dt} = 2 - \frac{y}{150}}$$

1st-order linear: $y' + \frac{1}{150}y = 2$

separable: $\frac{dy}{dt} = \frac{300-y}{150}$

Solve as 1st-order linear

$$y' + \frac{1}{150}y = 2$$

$$I = e^{\int \frac{1}{150}dt} = e^{\frac{1}{150}t}$$

$$e^{\frac{1}{150}t}y' + \frac{1}{150}e^{\frac{1}{150}t}y = 2e^{\frac{1}{150}t}$$

$$\frac{d}{dt}\left(e^{\frac{1}{150}t}y\right) = 2e^{\frac{1}{150}t}$$

$$e^{\frac{1}{150}t}y = \int 2e^{\frac{1}{150}t}\,dt = 300e^{\frac{1}{150}t} + C$$

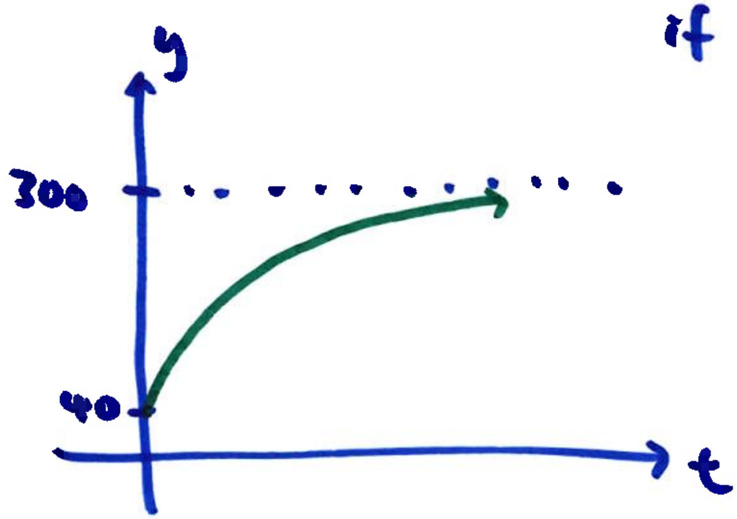
$$y = 300 + Ce^{-\frac{1}{150}t}$$

assume $y(0) = 40$ → initially there are 40 lb of salt in the tank

$40 = 300 + C$ → $C = -260$

$$\boxed{y(t) = 300 - 260e^{-\frac{1}{150}t}}$$ amt of salt in tank

$\lim_{t\to\infty} y = 300$ → salt conc. in tank → $\frac{1}{2}$ lb/gal if waited long enough



after a while, the tank is like a pipe because rate in = rate out example : same setup, but we let out 6 gal/min

rate in = $\left(\frac{1}{2} \text{ lb/gal}\right)(4 \text{ gal/min})$

rate out = $\underbrace{(\text{concentration})}(6 \text{ gal/min})$

assume initial volume is 600

net loss of 2 gal per min

volume = $600 - 2t$

concentration = $\dfrac{\text{amt of salt}}{\text{volume}} = \dfrac{y}{600-2t}$

$$\boxed{y' = 2 - \frac{6y}{600-2t} = 2 - \frac{3y}{300-t}}$$

linear? yes

separable?

$y' = \dfrac{600-2t-3y}{300-t}$ no!

$$y' + \frac{3}{300-t}y = 2$$

$$I = e^{\int \frac{3}{300-t}\,dt} = e^{-3\ln(300-t)} = e^{\ln(300-t)^{-3}}$$

$$= (300-t)^{-3} = \frac{1}{(300-t)^3}$$

$$y' + \frac{3}{300-t}\,y = 2$$

$$\frac{1}{(300-t)^3}\,y' + \frac{3}{300-t}\,\frac{1}{(300-t)^3}\,y = 2 \cdot \frac{1}{(300-t)^3}$$

$$\frac{d}{dt}\left(\frac{1}{(300-t)^3}\,y\right) = \frac{2}{(300-t)^3} = 2(300-t)^{-3}$$

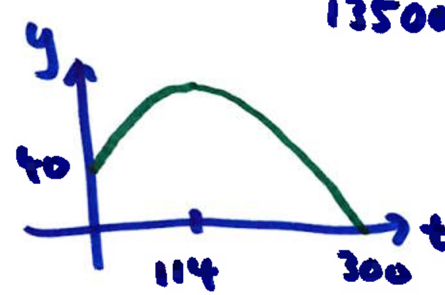
$$\frac{1}{(300-t)^3}\,y = \frac{1}{(300-t)^2} + C$$

$$\boxed{y = 300-t + C(300-t)^3}$$

$y(0)=40$

$\hookrightarrow C = -\frac{13}{1350000}$

$$\boxed{y = 300-t - \frac{13}{1350000}(300-t)^3}$$



max of y?

$y' = -1 + \frac{39}{1350000}(300-t)^2 = 0 \to$ solve for t: $t \approx 114$

$y_{max} = y(114) \approx 124$ lb

why?

$y' = 2 - \underbrace{\frac{3y}{300-t}}_{\text{rate out}}$

rate in

Sub in $t \approx 114$
$y \approx 124$

rate out ≈ 2

$t < 114 \to y' > 0$ rate in > rate out

$t > 114 \to y' < 0$ rate out wins

<u>example</u> Same set up, but rate of water out is 2 gal/min

same in: $(\frac{1}{2}\ \text{lb/gal})(4\ \text{gal/min})$

out: $\left(\frac{y}{\text{volume}}\right)(2\ \text{gal/min})$ volume $= 600 + 2t$

$$y' = 2 - \frac{2y}{600+2t} = 2 - \frac{y}{300+t}$$

$$y' + \frac{1}{300+t}\,y = 2 \qquad I = e^{\int \frac{1}{300+t}\,dt} = 300+t$$

$$\frac{d}{dt}\Big((300+t)y\Big) = 2(300+t)$$

$$\vdots$$

$$y = \frac{t^2 + 600t + 12000}{t+300}$$

valid up to the point when tank overflows