1.6 Substitution Methods and Exact Equation (part 1)

many equations are separable, linear, or both

many more are neither, for example $\frac{dy}{dx} = (x+y)^2$

Some can be turned into separable, linear, or both by using substitutions

<u>example</u> $\frac{dy}{dx} = (x+y)^2$

let $v = x+y$

transform eq. into one in v and x (no y)

from $v = x+y$

we get $\frac{dv}{dx} = \frac{d}{dx}x + \frac{d}{dx}y = 1 + \frac{dy}{dx}$

$$\frac{dy}{dx} = \underbrace{(x+y)}_{v}^2$$

$$\frac{dv}{dx} - 1$$

$$\frac{dv}{dx} = v^2 + 1 \qquad \text{separable in } v \text{ and } x$$

$$\frac{1}{v^2+1}\,dv = dx$$

integrate $\displaystyle \int \frac{1}{v^2+1}\,dv = \int \cancel{\frac{1}{x}}\,dx$

$$\tan^{-1}(v) = \cancel{\ln|x| + \ln|c|}$$

$$= x + C$$

$$v = \tan(x+c) = x+y$$

$$\boxed{y = \tan(x+c) - x}$$

any eq. in the form $\frac{dy}{dx} = f(ax+by+c)$ a, b, c constants

can be solved by using the sub $V = ax+by+c$

for example, $\frac{dy}{dx} = \sqrt{3x-2y+1} = f(3x-2y+1)$

Sub: $V = 3x-2y+1$

then proceed as in the previous example

A first-order eq. is said to be <u>homogeneous</u> if it can be written as $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

has different meanings in different contexts for example, $\frac{dy}{dx} = \frac{2x-y}{x+7y}$ not separable

not linear

if we divide top and bottom on right side by x

$$\frac{dy}{dx} = \frac{2-\left(\frac{y}{x}\right)}{1+7\left(\frac{y}{x}\right)} = f\left(\frac{y}{x}\right)$$

so eq. is homogeneous

$$\boxed{\text{homogeneous: use sub } v = \frac{y}{x}}$$

get v into eq. get y out

$$\frac{dy}{dx} = \frac{2-\left(\frac{y}{x}\right)}{1+7\left(\frac{y}{x}\right)}$$

$v = \frac{y}{x}$ $\qquad$ $\underline{y = vx}$

use this to find $\frac{dy}{dx}$ in terms of v and $\frac{dv}{dx}$

$y = vx$ ← function of x

must use product rule

$$\frac{dy}{dx} = v \cdot 1 + x \cdot \frac{dv}{dx} = v + x\frac{dv}{dx}$$

eq. becomes

$$v + x\frac{dv}{dx} = \frac{2-v}{1+7v}$$

$$x\frac{dv}{dx} = \frac{2-v}{1+7v} - v = \frac{2-v}{1+7v} - \frac{v(1+7v)}{1+7v}$$

$$= \frac{2-v-v-7v^2}{1+7v}$$

$$x\frac{dv}{dx} = \frac{2-2v-7v^2}{1+7v}$$

separable in v and x

$$\int \frac{1+7v}{2-2v-7v^2}\,dv = \int \frac{1}{x}\,dx$$

$$u = 2-2v-7v^2$$

$$du = (-2-14v)\,dv$$

$$= -2(1+7v)\,dv$$

$$-\frac{1}{2}\int \frac{1}{u}\,du = \int \frac{1}{x}\,dx$$

$$-\frac{1}{2}\ln u = \ln x + C$$

$$\ln u = -2\ln x + C$$

$$u = e^{-2\ln x + C} = e^{\ln x^{-2}} \cdot e^{C} = Cx^{-2}$$

$$2-2v-7v^2 = Cx^{-2}$$

$$2-2\left(\frac{y}{x}\right)-7\left(\frac{y}{x}\right)^2 = Cx^{-2}$$

make it prettier, multiply by x^2

$$\boxed{2x^2-2xy-7y^2 = C}$$

Bernoulli Differential Eq.

$$y' + P(x)y = Q(x)y^n \qquad n \neq 0,\ n \neq 1$$

eq. becomes linear

eq. becomes linear and/or separable

if $n=1$, $y' + Py = Qy$

$y' + \underbrace{(P-Q)}_{R}y = 0$ linear

$y' = -Ry$ might be separable

$\boxed{\text{for Bernoulli, use sub } v = y^{1-n}}$

<u>example</u>

$$x^2 y' + xy = y^2$$

make coefficient of y' 1

$$y' + \frac{1}{x} y = \frac{1}{x^2} y^{2} \rightarrow n = 2$$

let $v = y^{1-n} = y^{-1}$

rewrite eq. in terms of v and v', eliminating y and y'

from $v = y^{-1}$

$$v' = -y^{-2} y' \quad \text{so} \quad y' = -y^2 v'$$

back to $y' + \frac{1}{x} y = \frac{1}{x^2} y^2$

$$-y^2 v' + \frac{1}{x} y = \frac{1}{x^2} y^2$$

divide by $-y^2$

$$v' - \frac{1}{x} \cdot \boxed{\frac{1}{y}} = -\frac{1}{x^2}$$

$y^{-1} = v$

$$v' - \frac{1}{x} v = -\frac{1}{x^2}$$

linear in v and x

$$I = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

$$-x^{-1} v' + \frac{1}{x^2} v = -\frac{1}{x^3}$$

$$\frac{d}{dx}(x^{-1} v) = -\frac{1}{x^3}$$

$$x^{-1} v = \frac{1}{2x^2} + c$$

$$v = \frac{1}{2x} + cx \qquad v = y^{-1} = \frac{1}{y}$$

$$\frac{1}{y} = \frac{1}{2x} + cx$$

$$y = \frac{1}{\frac{1}{2x} + cx} = \frac{2x}{1 + 2cx^2} = \boxed{\frac{2x}{1 + cx^2}}$$