1.6 Substitution Methods and Exact Eq. (continued)

from calculus if $f(x,y) = c$ (level curve)

function of x

the total or exact differential of f is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0 \qquad f_x dx + f_y dy = 0$$

if $f(x,y)$ is a well-behaving function, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \quad \text{or} \quad f_{xy} = f_{yx}$$

if we see a differential eq. in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$ Exact Differential Eq.

and if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then this matches the form

$$\underbrace{\boxed{\frac{\partial f}{\partial x}}}_{M} dx + \underbrace{\boxed{\frac{\partial f}{\partial y}}}_{N} dy = 0$$

$$\frac{\partial}{\partial y}\left(\underbrace{\boxed{\frac{\partial f}{\partial x}}}_{M}\right) = \frac{\partial}{\partial x}\left(\underbrace{\boxed{\frac{\partial f}{\partial y}}}_{N}\right)$$

$f(x,y) = c$

MUST have an implicit solution $f(x,y) = c$

where $\frac{\partial f}{\partial x} = M \quad \frac{\partial f}{\partial y} = N$

example

$$\underbrace{(3x^2+2y^2)}_{M}dx + \underbrace{(4xy+6y^2)}_{N}dy = 0$$

Check if eq. is exact : $M_y = N_x$?

$$\left.\begin{aligned} \frac{\partial}{\partial y}M &= \frac{\partial}{\partial y}(3x^2+2y^2) = 4y \\ \frac{\partial}{\partial x}N &= \frac{\partial}{\partial x}(4xy+6y^2) = 4y \end{aligned}\right\} \text{match, so eq is exact}$$

because it is exact, it has an implicit solution

$$f(x,y) = C$$

where
$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= M \\ \frac{\partial f}{\partial y} &= N \end{aligned}\right\} \text{solve for } f$$

$$\frac{\partial f}{\partial x} = 3x^2 + 2y^2 \quad - \text{①}$$

$$\frac{\partial f}{\partial y} = 4xy + 6y^2 \quad - \text{②}$$

pick one of them to integrate, here, let's pick ①

from ①, $f = \underbrace{\int (3x^2 + 2y^2)\,dx}_{\substack{y \text{ is a constant} \\ \text{because this is} \\ \text{reversing } \frac{\partial}{\partial x}}} = x^3 + 2y^2 x + \underbrace{g(y)}_{\substack{\text{is constant} \\ \text{in } \frac{\partial}{\partial x}}} \quad - \text{③}$

we need to know $g(y)$

so take partial with respect to y of ③

and it should equal ②

from ③

$$\frac{\partial f}{\partial y} = 4yx + \frac{dg}{dy} = \underbrace{4xy + 6y^2}_{N \text{ from } ②}$$

so, $\frac{dg}{dy} = 6y^2$ $\quad g(y) = 2y^3 + c_1$

then $f(x,y) = x^3 + 2y^2x + 2y^3 + c_1$

solution is implicitly given as $f(x,y) = C$

$$\boxed{x^3 + 2y^2x + 2y^3 = C}$$

absorbed the c_1 from f

exact eqs often come as $M\,dx + N\,dy = 0$

but they don't have to be expressed that way

for example, $\frac{dy}{dx} = -\frac{y}{x}$

separable

homogeneous

linear

exact

$$x\,dy = -y\,dx$$

$$\underset{M}{y}\,dx + \underset{N}{x}\,dy = 0$$

$$\left.\begin{array}{l} M_y = 1 \\ N_x = 1 \end{array}\right\}$$ match so is exact

Some 2nd-order eqs can be solved as 1st-order using substitutions

first type: no x in the equation

$$y'' = -y$$

transform into a 1st-order w/ y as independent variable

let $p = \frac{dy}{dx} = y'$

then $y'' = \frac{dp}{d\boxed{x}} = \frac{dp}{d\boxed{y}} \overbrace{\frac{dy}{dx}}^{p} = p\frac{dp}{dy}$

independent variable

$$p\frac{dp}{dy} = -y$$

first order in p and y

separable

$$p\,dp = -y\,dy$$

$$\frac{1}{2}p^2 = -\frac{1}{2}y^2 + c$$

$$p^2 = -y^2 + c \nearrow \quad \frac{dy}{dx} = -\sqrt{y^2+c} \quad \text{separable}$$

$$p = \sqrt{-y^2+c} = \frac{dy}{dx} \rightarrow \cancel{y = \int p\,dx = \int \sqrt{-y^2+c}}$$

Second type : no y in the equation

keep x as independant variable

let $p = \frac{dy}{dx} = y'$

then $y'' = \frac{dp}{dx}$

$$xy'' + y' = x$$

$$x\frac{dp}{dx} + p = x$$ 1st order in p and x

$$\frac{dp}{dx} + \frac{1}{x}p = 1$$ linear

integrating factor $I = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

$$\frac{d}{dx}(xp) = x$$

$$xp = \frac{1}{2}x^2 + c_1$$

$$p = \frac{1}{2}x + c_1 x^{-1} = y'$$

$$y = \frac{1}{4}x^2 + c_1 \ln x + c_2$$