2.1 Population Models

$P(t)$: population as function of time

$P(t_0) = P_0$ initial population t_0 is often $t=0$

exponential growth/natural growth: $\frac{dP}{dt} = kP$ constant governing growth

when P is large → fast growth

when P is small → slow "

$\frac{dP}{dt} = kP$ $P(0) = P_0$ is separable

Solution: $P(t) = P_0 e^{kt}$ → doesn't limit how big P can get ($t \to \infty$, $P \to \infty$)

unrealistic since in reality population usually is limited by, for example, food. So there is a population cap.

tweak the right side of $\frac{dP}{dt} = kP$

change to $\frac{dP}{dt} = [\beta - \delta] P$ death parameter

birth parameter

both β and δ can depend on P

for example, $\underbrace{\beta = \beta_0 - \beta_1 P}$ β_0, β_1 positive constants

birth decreases linearly with P
(large $P \to$ slow birth)
one simple way to model environmental limitation

keep $\delta = \delta_0$ (constant)

plug into $\frac{dP}{dt} = [\beta - \delta] P$

$$\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta_0) P$$

$$= (\beta_0 - \delta_0) P - \beta_1 P^2 = \underbrace{aP}_{\text{birth rate}} - \underbrace{bP^2}_{\text{death rate}}$$

this is called <u>logistic equation</u>

$$= bP\left(\frac{a}{b} - P\right)$$

$$\boxed{\frac{dP}{dt} = kP(M - P)}$$

most common form of logistic eq.

M → carrying capacity → max pop the environment can sustain

notice $\frac{dP}{dt} = 0$ at $P = 0$ and $P = M$

when $0 \le P < M$, $\frac{dP}{dt} > 0$ (growth)

$P > M$, $\frac{dP}{dt} < 0$ (decline)

<u>example</u> A population is modeled by the logistic eq.

$$\frac{dp}{dt} = ap - bp^2 = kp(M-p)$$

the initial population is 120 $(t=0)$

there are 8 births per month and 6 deaths per month initially $(t=0)$

Find: carrying capacity

time to reach 95% of the carrying capacity

$$\frac{dp}{dt} = \underbrace{ap}_{\text{birth rate}} - \underbrace{bp^2}_{\text{death rate}}$$

at $t=0$, $p=120$

at $t=0$, 8 births/month $= aP = a(120) \rightarrow a = \frac{1}{15}$

at $t=0$, 6 deaths/month $= bP^2 = b(120)^2 \rightarrow b = \frac{1}{2400}$

$$\frac{dp}{dt} = \frac{1}{15}P - \frac{1}{2400}P^2 = \frac{1}{2400}P\left(\frac{1/15}{1/2400} - P\right)$$

$$= \frac{1}{2400}P(160 - P)$$

$KP(M-P)$

So, the carrying capacity is $\boxed{160}$

now let's solve $\frac{dp}{dt} = \frac{1}{2400}P(160-P)$ Separable Bernoulli

$$\frac{1}{P(160-P)}\,dp = \frac{1}{2400}\,dt$$

$$\underbrace{\int \frac{1}{P(160-P)}\,dp}_{\text{partial fractions}} = \int \frac{1}{2400}\,dt$$

$$\frac{1}{P(160-P)} = \frac{A}{P} + \frac{B}{160-P}$$

$$1 = A(160-P) + BP$$

$$0P + 1 = (B-A)P + 160A$$

$$B - A = 0 \qquad B = \frac{1}{160}$$

$$160A = 1 \rightarrow A = \frac{1}{160}$$

$$\int \frac{1}{P(160-P)}\,dP = \int \frac{1}{2400}\,dt$$

$$\int \left(\frac{1}{160}\cdot\frac{1}{P} + \frac{1}{160}\cdot\frac{1}{160-P}\right)dP = \int \frac{1}{2400}\,dt$$

$$\frac{1}{160}\ln P - \frac{1}{160}\ln(160-P) = \frac{1}{2400}t + C$$

$$\ln P - \ln(160-P) = \frac{1}{15}t + C$$

$$\ln\left(\frac{P}{160-P}\right) = \frac{1}{15}t + C$$

$$\frac{P}{160-P} = Ce^{\frac{1}{15}t}$$

$$P(0) = 120$$

$$\frac{120}{40} = Ce^{0} = C = 3$$

$$\frac{P}{160-P} = 3e^{\frac{1}{15}t}$$

$$P = (160-P)(3e^{\frac{1}{15}t})$$

$$P = 480e^{\frac{1}{15}t} - 3Pe^{\frac{1}{15}t}$$

$$P + 3Pe^{\frac{1}{15}t} = 480e^{\frac{1}{15}t}$$

$$\boxed{P = \frac{480e^{\frac{1}{15}t}}{1+3e^{\frac{1}{15}t}}}$$

carrying capacity : 160

95% of 160 = (0.95)(160)

$$(0.95)(160) = \frac{480e^{\frac{1}{15}t}}{1+3e^{\frac{1}{15}t}}$$ Solve for t

$\vdots$

$\boxed{t \approx 28}$ months