2.1 Population Models

population $P(t)$ with initial condition $P(0) = P_0$

the natural or exponential model is $\dfrac{dP}{dt} = kP$

grows/decays at a rate proportional to size.

$\dfrac{dP}{dt} = kP$ is separable, linear

$\dfrac{1}{P}\, dP = k\, dt$

$\ln P = kt + C$ $\qquad P = e^{kt+c} = e^{kt} \cdot e^{c} = Ce^{kt}$

$P(0) = P_0$

$P_0 = Ce^{0} = C$ $\qquad$ so, $\boxed{P(t) = P_0 e^{kt}}$

natural model is usually not good for long term modeling

this model does not account for the cap to the population (limited food/resources)

let's extend it a bit :

$$\frac{dP}{dt} = [\beta(P) - \delta(P)]P$$

birth parameter

death parameter

if β, δ are constants

then $\frac{dP}{dt} = (\beta - \delta)P$

$= kP$

natural model

let's model $\beta(t)$ as $\beta(P) = \underbrace{\beta_0 - \beta_1 P}$

birth decreases as P increases

keep death as $\delta = \delta_0$ (constant)

Sub into $\frac{dP}{dt} = (\beta - \delta)P$

$$= (\beta_0 - \beta_1 P - \delta_0)P$$

$$\frac{dP}{dt} = (\beta_0 - \delta_0)P - \beta_1 P^2$$

natural: $\frac{dP}{dt} = kP$

$$\frac{dP}{dt} = \underbrace{aP}_{\text{net birth rate}} - \underbrace{bP^2}_{\text{net death rate}} \qquad (a, b > 0)$$

this is called the <u>logistic equation</u>

$$\frac{dP}{dt} = bP\left(\frac{a}{b} - P\right)$$

$$\boxed{\frac{dP}{dt} = kP(M - P)}$$

carrying capacity
(equilibrium)

$\frac{dP}{dt} = 0$ at $P = 0$

$\frac{dP}{dt} = 0$ at $P = M$

if $P < M$ $\frac{dP}{dt} > 0$ (increasing)

if $P > M$ $\frac{dP}{dt} < 0$ (decreasing)

<u>example</u> A population of squirrels satisfies the logistic equation

$$\frac{dP}{dt} = aP - bP^2 = kP(M-P)$$

Initial population is 120

at $t=0$, there are 8 births per month and

6 deaths per month.

Find: carrying capacity

time to reach 95 % of the carrying capacity.

$$\frac{dP}{dt} = \underbrace{aP}_{\text{birth}} - \underbrace{bP^2}_{\text{death}}$$

at $t=0$, $P=120$, birth rate $= aP = 120a = 8$

$$a = \frac{1}{15}$$

at $t=0$, $P=120$. death rate $= bP^2 = (120)^2 b = 6$

$$b = \frac{1}{2400}$$

so. $$\frac{dP}{dt} = \frac{1}{15}P - \frac{1}{2400}P^2$$

$$= \boxed{\frac{1}{2400}P(160-P)} = kP(M-P)$$

carrying capacity is 160

to find time to reach 95% of 160, we need

$P(t)$: solve $$\frac{dP}{dt} = \underbrace{\frac{1}{15}P - \frac{1}{2400}P^2}_{\text{linear? NO.}} = \underbrace{\frac{1}{2400}P(160-P)}_{\text{separable}}$$

linear? NO.

$P' + A(t)P = B(t)$

Bernoulli? Yes

$P' + A(t)P = B(t)P^n$

let's solve it as a separable

$$\frac{1}{P(160-P)}\,dP = \frac{1}{2400}\,dt$$

$$\underbrace{\int \frac{1}{P(160-P)}\,dP}_{\text{partial fraction expansion}} = \int \frac{1}{2400}\,dt$$

$$\frac{1}{P(160-P)} = \frac{A}{P} + \frac{B}{160-P}$$

$$1 = A(160-P) + BP$$

$$0P + 1 = (B-A)P + 160A$$

$$B-A=0$$

$$160A = 1$$

$$A = \frac{1}{160}$$

$$B = \frac{1}{160}$$

$$\int\left(\frac{1}{160}\cdot\frac{1}{P} + \frac{1}{160}\cdot\frac{1}{160-P}\right)dP = \int\frac{1}{2400}\,dt$$

$$\frac{1}{160}\ln P - \frac{1}{160}\ln(160-P) = \frac{1}{2400}t + C$$

$$\ln P - \ln(160-P) = \frac{1}{15}t + C$$

$$\ln\left(\frac{P}{160-P}\right) = \frac{1}{15}t + C$$

$$\frac{P}{160-P} = Ce^{\frac{1}{15}t} \qquad P(0) = 120$$

$$\frac{120}{40} = C = 3$$

$$\frac{P}{160-P} = 3e^{\frac{1}{15}t}$$

$$P = (160-P)(3e^{\frac{1}{15}t})$$

$$= 480e^{\frac{1}{15}t} - 3e^{\frac{1}{15}t}P$$

$$(1+3e^{\frac{1}{15}t})P = 480e^{\frac{1}{15}t}$$

$$P = \frac{480e^{\frac{1}{15}t}}{1+3e^{\frac{1}{15}t}} = \boxed{\frac{480}{e^{-\frac{1}{15}t}+3}}$$

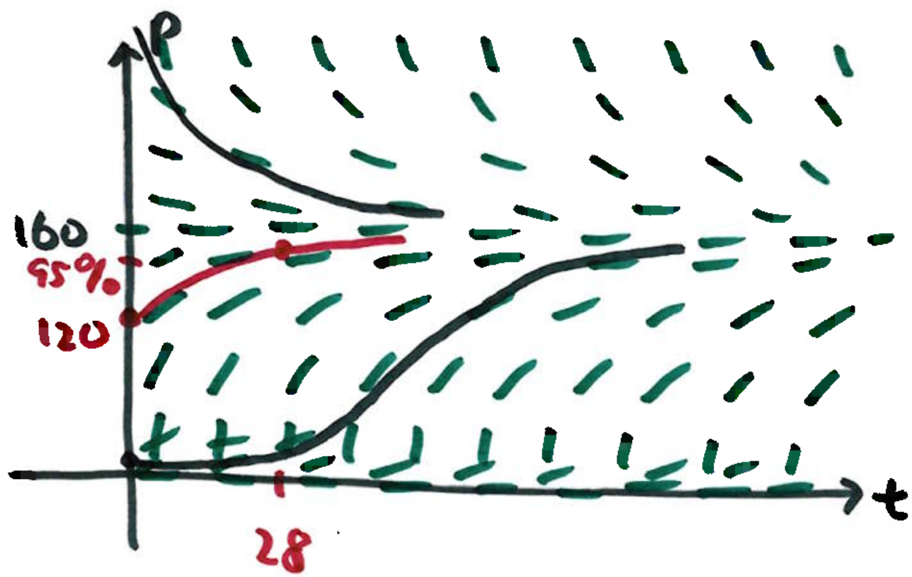
time to reach 95% of 160 → t = ? when $P = (0.95)(160)$

$$(0.95)(160) = \frac{480}{e^{-\frac{1}{15}t} + 3}$$

$t \approx 28$ months

$$\frac{dP}{dt} = \frac{1}{2400} P(160 - P)$$

$P' = 0$ when $P = 0$, $P = 160$



$P < 160$, $P' > 0$

steeper farther away from 160

$P > 160$, $P' < 0$

steeper farther away