1.1 Intro to Differential Equations

an equation that contains a derivative

e.g. $\dfrac{dy}{dx} = \cos x$

more generally, $\dfrac{dy}{dx} = f(x,y)$ e.g. $y' = y$

applications: Newton's Law: $F = ma$ second deriv. of position

$F = m x''$

$$\frac{d^2r}{dt^2} = -\frac{Gm}{r^2}$$ Newton's Law of gravitation

Earth mass m

$$\frac{dP}{dt} = K(L-P)$$ P(t) is population

L : pop. capacity

goal: solve the differential eq.

for example, $y' = y$ "solve" $\rightarrow$ $y = ?$

$\rightarrow$ function whose deriv. is itself

$y = Ae^x$ A: constant

if something is a solution to a differential eq. $\frac{dy}{dx} = f(x,y)$

then it <u>must</u> satisfy the differential eq.

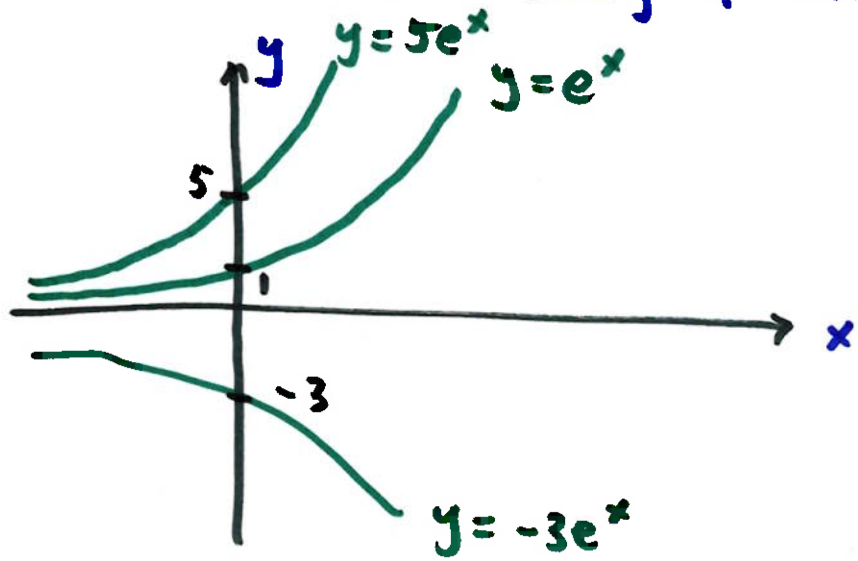
for example, $y' = y$ with $y = Ae^x$ as a solution

it satisfies the $y' = y$ because

$\underbrace{y}_{\text{right side of } y'=y} = Ae^x$ $\quad$ $\underbrace{y'}_{\text{left side of } y'=y} = Ae^x$ $\rightarrow$ so $y = Ae^x$ is a solution usually we use "C" for the constant

$y' = y$ has $\underbrace{y = Ce^x}$ as a solution

includes a family of solutions



$y = Ce^x$ is often called the general solution if we don't specify C

if a C is know, then it's a particular solution

the intersection at $x = 0 \to \underbrace{y(0) = \text{constant}}_{\text{initial condition}}$

the highest derivative in a differential eq. is called the <u>order</u> of the differential eq.

$y' = y$ is a 1st-order

$r'' = -\frac{GM}{r}$ is 2nd-order

$e^y y' = 1$ is 1st-order

<u>example</u> Find all values of r such that $y = e^{rx}$ is a solution to $y'' + y' - 2y = 0$

sub in $y = e^{rx}$ $\quad y' = re^{rx}$ $\quad y'' = r^2 e^{rx}$

$r^2 e^{rx} + re^{rx} - 2e^{rx} = 0$

$e^{rx}(r^2 + r - 2) = 0$

$e^{rx} \neq 0$ for any r so $r^2 + r - 2 = 0$

$(r+2)(r-1) = 0 \quad r = -2, r = 1$

solutions are : $y = e^{-2x}$ and $y = e^x$

1.2 Integrals as General and Particular Solutions

first-order: $\dfrac{dy}{dx} = \underbrace{f(x,y)}$

generally, can contain x and y

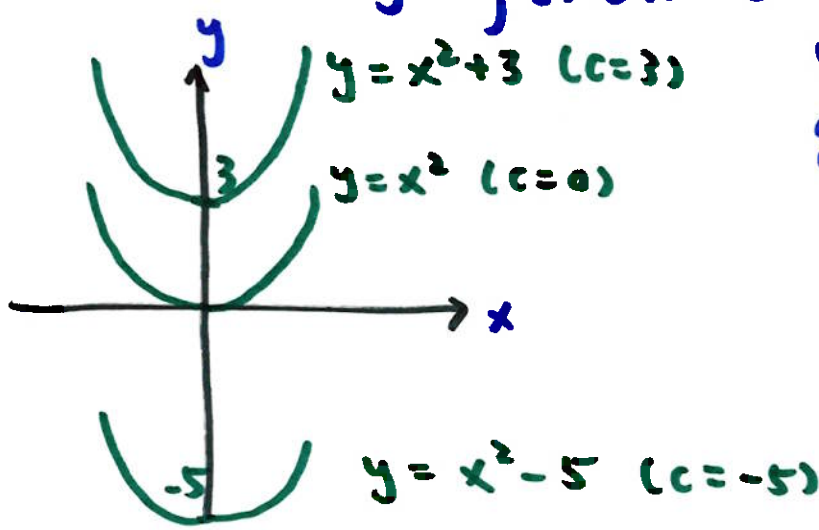
how we solve depends on what f looks like

today: simplest kind → f only depends on x

for example, $\dfrac{dy}{dx} = 2x$ → we know how to solve by calculus!

$$y = \int 2x\,dx = \underbrace{x^2 + C}$$

general solution because C is not specified



if we know C, then we have a particular solution

usually determined by a point the curve goes through

for example, $(1, 3) \to y(1) = 3$ (initial or side condition)

$$y = x^2 + C$$

$$3 = (1)^2 + C \to C = 2 \quad \text{so} \quad y = x^2 + 2$$ (particular solution)

<u>example</u> solve $\frac{dy}{dx} = x\sqrt{x^2+1}$ $\quad y(0) = 2$ initial-value problem (IVP)

first, find y: $\quad y = \int x\sqrt{x^2+1}\,dx$

substitution: $u = x^2 + 1$

$du = 2x\,dx$

$$y = \int \tfrac{1}{2} u^{1/2}\,du$$

$$= \frac{1}{2} \frac{u^{3/2}}{3/2} + C = \frac{1}{3}(x^2+1)^{3/2} + C$$

(general)

use $y(0)=2$ to find C

$y = \frac{1}{3}(x^2+1)^{3/2} + C$

$2 = \frac{1}{3}(1)^{3/2} + C$ so $C = 5/3$

$$\boxed{y = \frac{1}{3}(x^2+1)^{3/2} + \frac{5}{3}}$$

<u>example</u>

$$\frac{dy}{dx} = \frac{1}{x^2+1}$$

$$y = \int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$$

no initial condition so we only have the general solution

alternative: trig sub

$F = ma = m\dfrac{d^2x}{dt^2}$ Solution: $x(t)$

integrate once: velocity $v(t) = x'(t)$

integrate again: position $x(t)$

example Solve $\dfrac{d^2x}{dt^2} = -10$ $\underbrace{\overbrace{x'(0) = 20}^{\text{initial velocity}}, \overbrace{x(0) = 5}^{\text{initial position}}}_{\text{2nd-order} \to \text{Two initial conditions to solve completely}}$

$x'' = -10$

$x' = \int -10\,dt = -10t + C_1$ find C_1: $x'(0) = 20$

$20 = -10(0) + C_1$

so $C_1 = 20$

$x' = -10t + 20$

$x = \int(-10t + 20)\,dt = -5t^2 + 20t + C_2$ find C_2: $5 = -5(0) + 20(0) + C_2$

so $C_2 = 5$

$\boxed{x = -5t^2 + 20t + 5}$