2.4 Numerical Approximation: Euler's Method

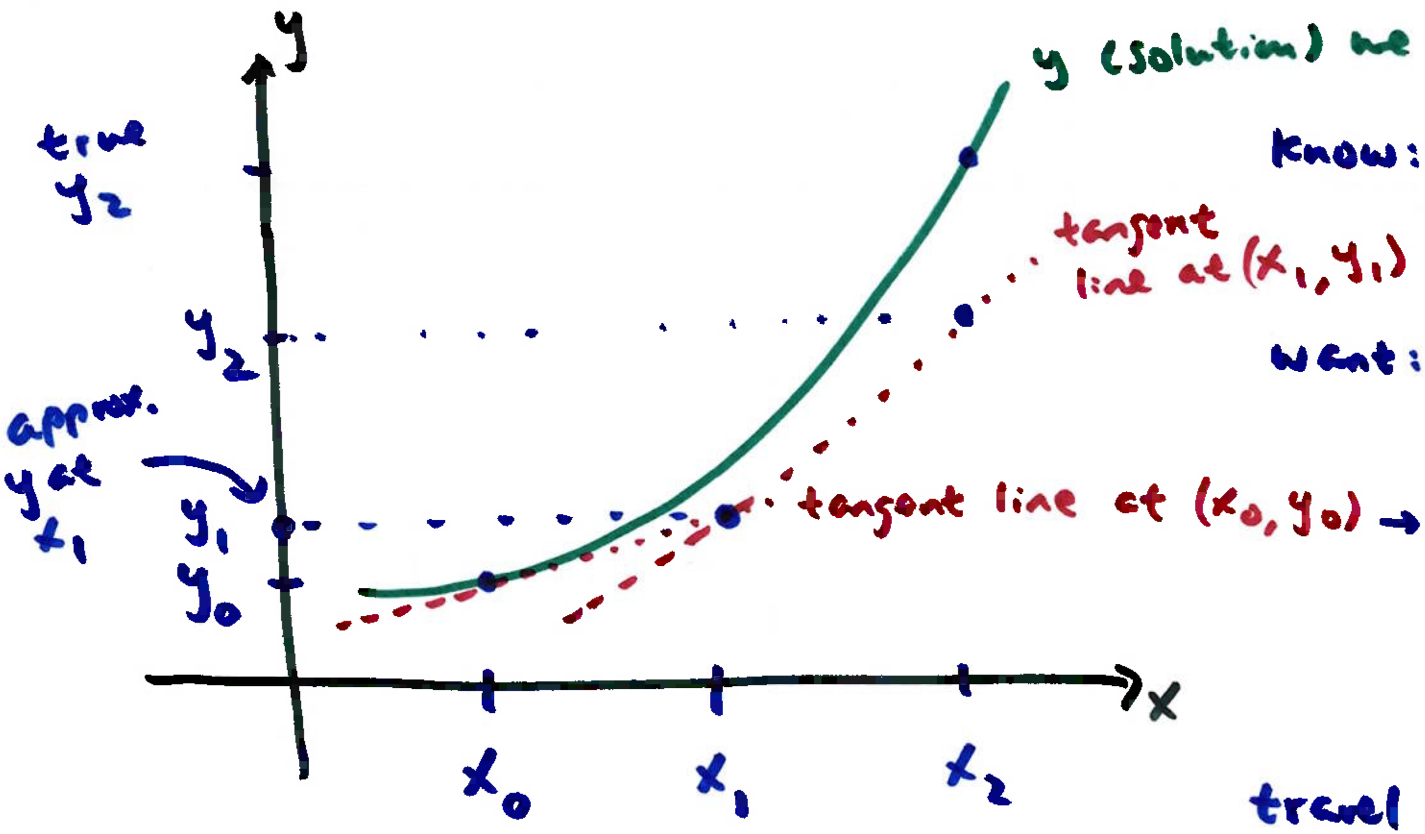
we know how to solve: separable
linear
homogeneous
Bernoulli
Exact

if an equation that is not one or the above or if it is impractical to ~~so~~ solve, we can use numerical methods (computers use these)

Euler's method (tangent line method)

basic idea: use successive linearization

$y' = \underbrace{f(x,y)}_{\text{derivative of } y \text{ at } (x,y)}$ goal: find y



Know: initial condition $y(x_0) = y_0$

want: y at some other x

→ near $x = x_0$, y on the true curve $\approx$ y on the tangent line

travel on tangent line at (x_0, y_0), get to $y_1 \approx$ "real" y at x_1

repeat the process until $x =$ target value

outline of Euler's method

x_0 : given $\qquad\qquad y' = f(x, y)$

y_0 : given

decide step size : h (change in x)

$x_1 = x_0 + h$ new x

tangent line at (x_0, y_0): $y - y_0 = f'(x_0, y_0)(x - x_0)$

$$y = y_0 + f'(x_0, y_0)(\underbrace{x - x_0}_{h})$$

estimate of y at $x = x_1$

$$y_1 = y_0 + f'(x_0, y_0)(x - x_0)$$

$$\boxed{y_1 = y_0 + f'(x_0, y_0)\, h}$$

if x_1 is not the target x, repeat

in general,

$$\boxed{\begin{aligned} x_{n+1} &= x_n + h \\ y_{n+1} &= y_n + f'(x_n, y_n)\, h \end{aligned}}$$

<u>example</u> $\quad y' = \underbrace{2y-3x}_{f(x_0,y_0)} \qquad y(0)=1$

estimate $y(0.5)$ using step size of $h = 0.25$

$x_0 = 0$

$y_0 = 1$

$x_1 = x_0 + h = 0 + 0.25 = 0.25$

$y_1 = y_0 + f(x_0, y_0)h$

$= 1 + [2(\underset{y_0}{1}) - 3(\underset{x_0}{0})](0.25) = 1.5$

$x_1 \neq$ target x, so repeat

$x_2 = x_1 + h = 0.25 + 0.25 = 0.5$

$y_2 = y_1 + f(x_1, y_1)h$

$= 1.5 + [2(1.5) - 3(0.25)](0.25)$

$= \boxed{2.0625}$ stop, since $x_2 =$ target x

our estimate of $y(0.5)$ where $y' = 2y - 3x$, $y(0) = 1$

in this case, since $y' = 2y - 3x$ is linear, we can solve it and compare to the estimate (normally not the case)

$$y' - 2y = -3x \qquad y(0) = 1$$

$$I = e^{\int -2\,dx} = e^{-2x}$$

$$\frac{d}{dx}\left(e^{-2x} y\right) = -3x e^{-2x}$$

$$e^{-2x} y = \int -3x e^{-2x}\,dx \quad \text{(by parts)}$$

$$\vdots$$

$$y = \frac{3}{4}(2x+1) + \frac{1}{4} e^{2x}$$

<u>true</u> $y(0.5) = 2.1796$

estimate from Euler's method w/ $h = 0.25$ is $y(0.5) = 2.0625$

if error is not acceptable, decrease step size (smaller h)

$\rightarrow$ more steps

if we had used $h = 0.01$ instead, $y(0.5) \approx 2.1729$

in general, we don't know the true y value, so how do we know if our estimate is good?

→ if estimate is good, then further refinement of step size (smaller h) will not improve or change the estimate significantly

in $y' = 2y - 3x$, $y(0) = 1$ estimate $y(1)$

$h = 0.5$ $y(1) = 3.25$

$h = 0.05$ $y(1) = 3.932$

} big change so estimate improved by smaller h

$h = 0.01$ $y(1) = 4.061$

} still changing

$h = 0.001$ $y(1) = 4.094$

$h = 0.0005$ $y(1) = 4.095$

} not changing much
this is probably very close to true $y(1)$

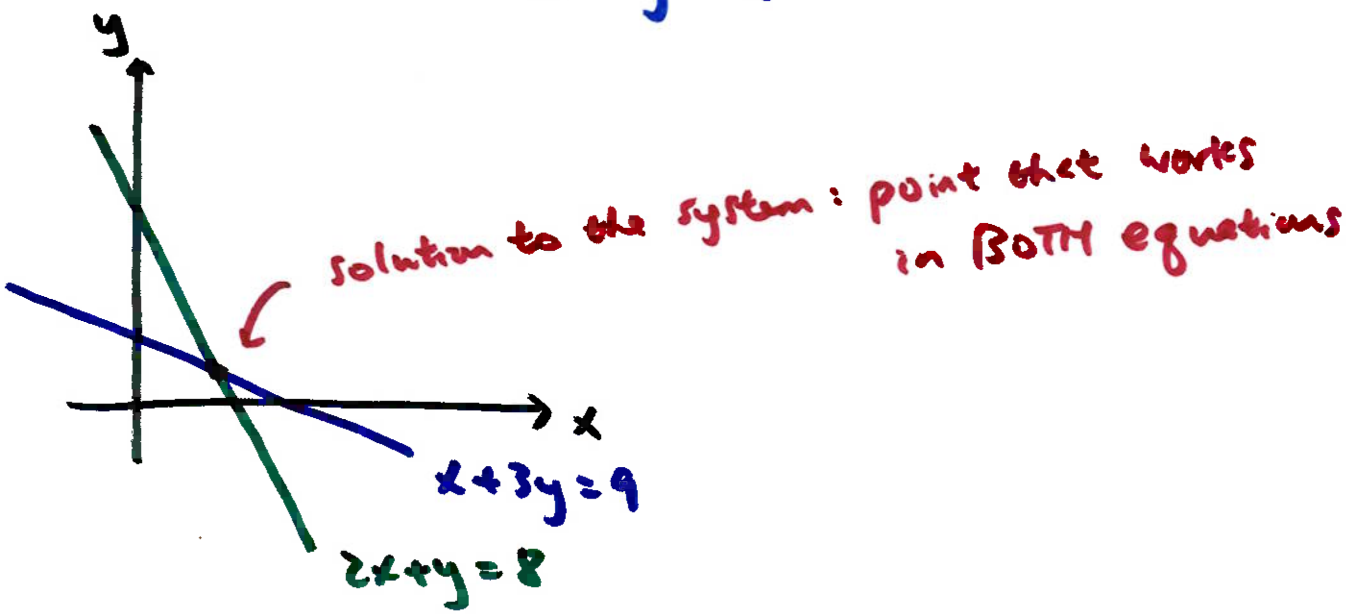
3.1 Introduction to Linear Systems

System of two variables:
$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

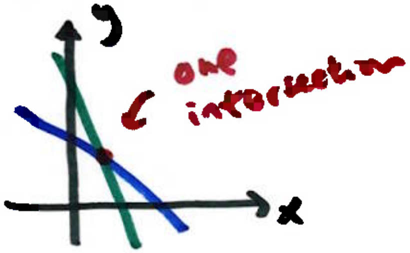
find all (x,y) that satisfy the system

$a_1 x + b_1 y = c_1$ and $a_2 x + b_2 y = c_2$ are <u>lines</u>

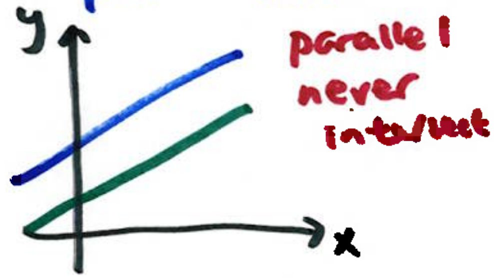
for example,
$$x + 3y = 9$$
$$2x + y = 8$$



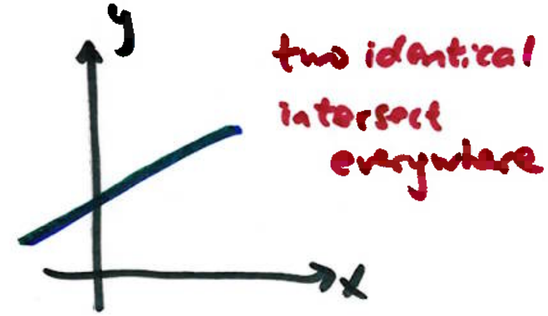
given two lines: three possibilities



one solution (consistent system)



no solution (inconsistent)



infinitely many (consistent system)

many ways to solve : first is method of elimination

<u>example</u>

$$x + 3y = 9 \quad - \ (1)$$

$$2x + y = 8 \quad - \ (2)$$

multiply either by a constant then add to the other such that one variable is eliminated

multiply (1) by -2 $\quad -2x - 6y = -18 \quad - \ (3)$

add (2) and (3)

$$-5y = -10 \rightarrow \boxed{y = 2}$$

then use (1), (2) or (3) to find x : $\boxed{x = 3}$

<u>example</u>

$$x + 2y = 4 \quad - (1)$$
$$2x + 4y = 9 \quad - (2)$$

multiply (1) by -2, add to (2)

$0 = 1$ obviously wrong

→ no solution

→ parallel lines

<u>example</u>

$$x + 2y = 4 \quad - (1)$$
$$2x + 4y = 8 \quad - (2)$$

$-2(1) + (2) \rightarrow 0 = 0$ true!

→ ~~any~~ many x and y work

Solutions: pick x, then $y = \frac{4-x}{2}$

$$\left(x, 2 - \tfrac{1}{2}x \right)$$

two identical lines

3D system : $a_1 x + b_1 y + c_1 z = d_1$

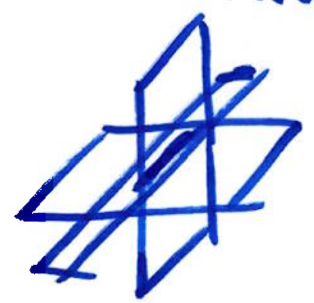
$a_2 x + b_2 y + c_2 z = d_2$

$a_3 x + b_3 y + c_3 z = d_3$

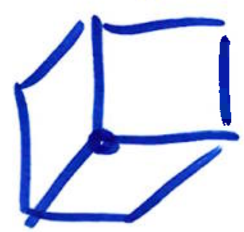
3 planes

↳ infinitely many (same planes)

" " (intersect at a line)



one point



no solution (all parallel)

(two meet but 3rd one doesn't)

use the same elimination method as in 2D example

$$x + 5y + z = 2 \quad - ①$$
$$2x + y - 2z = 1 \quad - ②$$
$$x + 7y + 2z = 3 \quad - ③$$

work w/ two at a time to eliminate at least one variable

$$\left.\begin{array}{ll} -2① + ② & -9y - 4z = -3 \quad - ④ \\ -① + ③ & 2y + z = 1 \quad - ⑤ \end{array}\right\} \text{2D sys.}$$

$$4⑤ + ④ \quad -y = 1 \quad \boxed{y = -1}$$

from ⑤ $\quad z = 1 - 2y = 1 - 2(-1) = 3$

$$\boxed{z = 3}$$

from ① $\quad x = 2 - 5y - z = 4 \quad \boxed{x = 4}$

one solution