3.2 Matrices and Gaussian Elimination

last time:

$$x_1 + 3x_2 = 9$$
$$2x_1 + x_2 = 8$$

we used the method of elimination to find $x_1 = 3$, $x_2 = 2$

the method can be made more compact and organized by using the <u>coefficient matrix</u>

$$\begin{matrix} x_1 + 3x_2 = 9 \\ 2x_1 + x_2 = 8 \end{matrix} \quad \rightarrow \quad \begin{array}{c} \begin{matrix} x_1 & x_2 & \end{matrix} \\ \begin{bmatrix} 1 & 3 & 9 \\ 2 & 1 & 8 \end{bmatrix} \end{array}$$

<u>augmented</u> matrix (include the right side)

matrix: array of numbers (here the numbers are coefficients)

this matrix has <u>two</u> rows and <u>three</u> columns

→ this is a 2×3 matrix ("two by three") row: equation   column: variable

method of elimination consists of elementary row operations

- swap two rows
- multiply one row by a number, then add to another row
- multiply one row by a (non-zero) number

back to $\begin{bmatrix} 1 & 3 & 9 \\ 2 & 1 & 8 \end{bmatrix}$

$$\xrightarrow{(-2)R_1+R_2} \begin{bmatrix} 1 & 3 & 9 \\ 0 & -5 & -10 \end{bmatrix}$$

$$\xrightarrow{(-\frac{1}{5})R_2} \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 2 \end{bmatrix}$$

this is now in <u>row echelon</u> form

row echelon form: a row of all zeros is at the bottom

the leading entry (the left most non-zero number) of each row is to the right of the leading entry of the row above

→ "pivot"

row 1 leading try

$$\begin{bmatrix} \boxed{1} & 3 & 9 \\ 0 & \boxed{1} & 2 \end{bmatrix}$$

pivot of row 2

row 1 : pivot is 1 (in column 1)

row 2 : pivot is 1 (in column 2) to the right of row above

from here, we can solve for x_1 and x_2

$$\begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 2 \end{bmatrix}$$

row 2 : $0 \cdot x_1 + 1 \cdot x_2 = 2 \rightarrow \boxed{x_2 = 2}$

row 1: $1 \cdot x_1 + 3 \cdot x_2 = 9 \rightarrow x_1 = -3x_2 + 9$

$= -3(2) + 9 = 3$

$\boxed{x_1 = 3}$

this method is called <u>Gaussian elimination</u>

(or row reduction)

<u>example</u>

$$x_2 + 4x_3 = -3$$
$$x_1 + 3x_2 + 6x_3 = 4$$
$$2x_1 + 5x_2 + 8x_3 = 5$$

put into augmented matrix

$$\begin{bmatrix} 0 & 1 & 4 & -3 \\ 1 & 3 & 6 & 4 \\ 2 & 5 & 8 & 5 \end{bmatrix}$$

put into row echelon form

if possible, have a "1" in 1st row in column 1

$$\xrightarrow{\text{swap}(R_1,R_2)} \begin{bmatrix} 1 & 3 & 6 & 4 \\ 0 & 1 & 4 & -3 \\ 2 & 5 & 8 & 5 \end{bmatrix}$$

$$\xrightarrow{(-2)R_1+R_3} \begin{bmatrix} 1 & 3 & 6 & 4 \\ 0 & 1 & 4 & -3 \\ 0 & -1 & -4 & -3 \end{bmatrix}$$

$$\xrightarrow{R_2+R_3} \begin{bmatrix} 1 & 3 & 6 & 4 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$ row echelon form

from row echelon, solve from bottom up

row 3: $0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 = -6$

$0 = -6$ (false, indicating <u>no</u> solution)

system is inconsistent

<u>example</u>

$$x_1 - x_2 + x_3 = 7$$
$$3x_1 + 2x_2 - 12x_3 = 11$$
$$4x_1 + x_2 - 11x_3 = 18$$

← pivot of row 1, use row ops to make numbers below zero

$$\begin{bmatrix} \boxed{1} & -1 & 1 & 7 \\ 3 & 2 & -12 & 11 \\ 4 & 1 & -11 & 18 \end{bmatrix}$$

$$\xrightarrow{(-3)R_1+R_2} \begin{bmatrix} 1 & -1 & 1 & 7 \\ 0 & 5 & -15 & -10 \\ 4 & 1 & -11 & 18 \end{bmatrix}$$

pivot, #s below are zero

$$\xrightarrow{(-4)R_1+R_3} \begin{bmatrix} \boxed{1} & -1 & 1 & 7 \\ 0 & \boxed{5} & -15 & -10 \\ 0 & 5 & -15 & -10 \end{bmatrix}$$

now work on this pivot &
(make # below zero)

$$\xrightarrow{(-1)R_2+R_3} \begin{bmatrix} 1 & -1 & 1 & 7 \\ 0 & 5 & -15 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

row echelon form

read bottom up to solve

row 3: $0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 = 0 \rightarrow 0 = 0$

(infinitely-many solutions)

one of the three variables is arbitrary ("free") we pick the one without pivot in its column to be free (others are basic variables)

here, x_3 doesn't have a pivot in its column

so, x_3 is free $\rightarrow$ $x_3 = t$

row 2: $5x_2 - 15x_3 = -10 \rightarrow x_2 = 3x_3 - 2 = 3t - 2$

row 1: $x_1 - x_2 + x_3 = 7$

$x_1 = x_2 - x_3 + 7$

$= (3t-2) - t + 7 = 2t + 5$

solution: $(2t+5, 3t-2, t)$ t: some number

3.3 Reduced Row Echelon Form

row echelon : make numbers below a pivot zero
don't care what's above

reduce row echelon: make the numbers above zero, too
(goal: pivot the only non-zero in its column)
and make pivot 1

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

pivot

don't care what's above pivot

$$\xrightarrow{(-3)R_1+R_2} \begin{bmatrix} \boxed{1} & 2 \\ 0 & \boxed{-2} \end{bmatrix}$$ pivot

row echelon

0 below pivot

$$\xrightarrow{R_2+R_1} \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{-2} \end{bmatrix}$$

pivots

reduced row echelon

$$\xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \end{bmatrix}$$

pivots are 1

row echelon is NOT unique

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{swap(R_1,R_2)} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{3}(R_1)+R_2} \begin{bmatrix} 3 & 4 \\ 0 & 2/3 \end{bmatrix}$$ row echlon

but <u>reduced</u> row echelon is unique

$$\xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & 4/3 \\ 0 & 2/3 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & 0 \\ 0 & 2/3 \end{bmatrix}$$

$$\xrightarrow{\frac{3}{2}R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ same reduced row echelon

solving system : reduced row echlon doesn't require back substitution

<u>example</u>

$$x_1 + 3x_2 + 2x_3 = 12$$
$$2x_1 + 5x_2 + 2x_3 = 12$$
$$2x_1 + 7x_2 + 7x_3 = 43$$

$$\begin{bmatrix} 1 & 3 & 2 & 12 \\ 2 & 5 & 2 & 12 \\ 2 & 7 & 7 & 43 \end{bmatrix}$$

$$\xrightarrow[-2R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 3 & 2 & 12 \\ 0 & -1 & -2 & -12 \\ 0 & 1 & 3 & 19 \end{bmatrix}$$

$$\xrightarrow[+3R_2+R_1]{R_2+R_3} \begin{bmatrix} 1 & 0 & -4 & -24 \\ 0 & -1 & -2 & -12 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\xrightarrow{(-1)R_2} \begin{bmatrix} 1 & 0 & -4 & -24 \\ 0 & 1 & 2 & 12 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\xrightarrow[(4)R_3+R_1]{(-2)R_3+R_2} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 7 \end{bmatrix} \quad \text{reduced row echelon}$$

$$\left.\begin{array}{ll} \text{row 3:} & x_3 = 7 \\ \text{row 2:} & x_2 = -2 \\ \text{row 1:} & x_1 = 4 \end{array}\right\} \text{don't require back substitution}$$

the process to ~~solve~~ put into reduced row echlon is called Gauss-Jorden Elimination each all zero row corresponds to **one** free-variable

(free variables: variables w/o pivot in its column)

after reduction →

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \text{right side} \\ \end{array}$$
$$\left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

} two all zero rows

Two free variables

$\underbrace{x_2,\ x_3}_{\text{free}}$ don't have pivots

$x_3 = t$

$x_2 = r$

t, r are real numbers

row 1: $x_1 + 2x_2 + 3x_3 = 4$

$x_1 = -2r - 3t + 4$