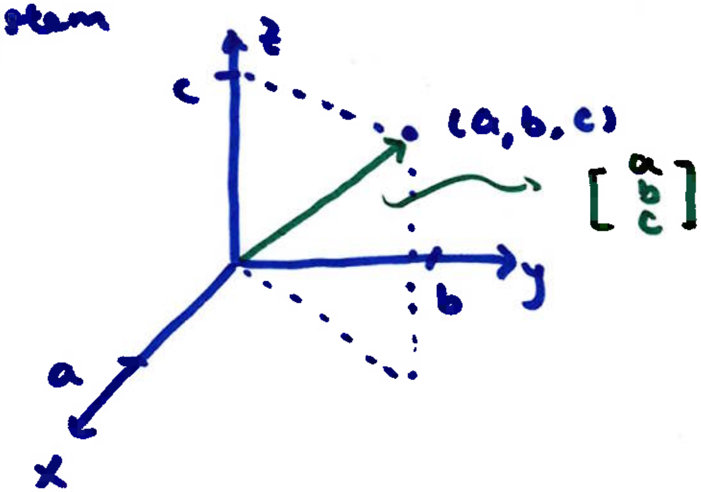
4.1 The Vector Space R^3

3D coordinate system



all (a,b,c) points are contained in this space

each point (a,b,c) can also be seen as the tip of a vector w/ tail at origin

so each (a,b,c) is equivalent to vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

the space that contains all possible 3-component vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is called the vector space R^3

inside this vector space the familiar operations of addition and scalar multiplication hold

$\vec{a} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} 4 \\ 6 \\ 5 \end{bmatrix}$ $\quad \vec{a}+\vec{b} = \begin{bmatrix} 5 \\ 9 \\ 7 \end{bmatrix}$ → also a part of vector space R^3

$3\vec{a} = \begin{bmatrix} 3 \\ 9 \\ 6 \end{bmatrix}$ also a member of R^3

two vectors $\vec{u}$ and $\vec{v}$ are said to be linearly dependent if one is a scalar multiple of the other

$$\vec{u} = c\vec{v} \quad \text{or} \quad \vec{v} = d\vec{u}$$

for example, $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ clearly $\vec{v} = -2\vec{u}$

so $\vec{u}, \vec{v}$ are linearly dependent

if we cannot express $\vec{u} = c\vec{v}$ or $\vec{v} = d\vec{u}$, then they are linearly independent

$\boxed{\text{linearly independent} \rightarrow a\vec{u} + b\vec{v} = \vec{0} \text{ implies } a = b = 0}$

→ the only way to get $\vec{0}$ is if we multiply both by 0

if not, then $a\vec{u} + b\vec{v} = \vec{0} \rightarrow \vec{v} = -\frac{a}{b}\vec{u} = c\vec{u}$

$\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ clearly not multiple of each other so clearly independent

if we try to solve $a\vec{u} + b\vec{v} = \vec{0}$

$$a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\vec{b}}$$

$$\det\Bigg(\underbrace{\begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}}_{\text{vectors } \vec{u}, \vec{v} \text{ as columns}}\Bigg) = -4 \neq 0$$

$\boxed{\text{if } \vec{u} \text{ and } \vec{v} \text{ are linearly independent, then } \det A \text{ (where A's columns are } \vec{u} \text{ and } \vec{v}) \text{ is nonzero}}$

same system solved using Gaussian elimination

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & -2 & 0 \end{bmatrix} \longrightarrow \begin{array}{ccc} a & b & \\ \hline \end{array}$$

$$\begin{array}{c} \\ \end{array}\begin{array}{ccc} a & b & \\ \end{array}$$

$$\begin{bmatrix} \boxed{1} & 2 & 0 \\ 0 & \boxed{-4} & 0 \end{bmatrix}$$

two pivots, two variables $\rightarrow$ unique solution

$A\vec{x} = \vec{b}$ has unique solution if $\det A \neq 0 \rightarrow$ if columns of A are linearly independant

$A\vec{x} = \vec{b}$ if A is 2×2 for simplicity (but true for $n \times n$)

$$\vec{x} = A^{-1}\vec{b} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\vec{b}$$ if $\det A \neq 0$, A^{-1} exists

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

add one more : $A\vec{x} = \vec{b}$ has unique solution $\Leftrightarrow$ cols of A are linearly indp $\leftrightarrow$ $\det A \neq 0$ $\leftrightarrow$ A^{-1} exists linearly indp : only way to get $\vec{0}$ is if each multiplied by 0

$$\vec{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 7 \\ 8 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

questions: are they linearly independent?

if not, find one as a <u>linear combination</u> of the other two

if linearly <u>indp</u>: $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$ if and only if $a = b = c$

linearly <u>dep</u> equivalent to $c_1\vec{u} + c_2\vec{v} = \vec{w}$ for some c_1, c_2 not all zero

↳ solve it this way: if c_1, c_2 not both zero, then the vectors are dependent and $c_1\vec{u} + c_2\vec{v} = \vec{w}$ gives us the answer to the 2nd question

$$c_1\begin{bmatrix}4\\3\end{bmatrix}+c_2\begin{bmatrix}7\\8\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}$$

$$\begin{bmatrix}4&7\\3&8\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}4&7\\3&8\end{bmatrix}^{-1}\begin{bmatrix}-1\\2\end{bmatrix}$$

$$=\frac{1}{11}\begin{bmatrix}8&-7\\-3&4\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}$$

$$=\frac{1}{11}\begin{bmatrix}-22\\11\end{bmatrix}=\begin{bmatrix}-2\\1\end{bmatrix}=\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

which means: $\underbrace{-2\underset{\vec{u}}{\begin{bmatrix}4\\3\end{bmatrix}}+1\underset{\vec{v}}{\begin{bmatrix}7\\8\end{bmatrix}}}_{\text{linear combination}}=\underset{\vec{w}}{\begin{bmatrix}-1\\2\end{bmatrix}}$

we can express one as linear combo of the other two

$\longrightarrow$ linearly dependent

__example__

$\vec{u} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} -5 \\ 1 \\ -1 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 0 \\ -4 \\ -1 \end{bmatrix}$

linearly independent?

$\rightarrow a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$ if and only if $a = b = c = 0$

$$a \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} -5 \\ 1 \\ -1 \end{bmatrix} + c \begin{bmatrix} 0 \\ -4 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 4 & -5 & 0 \\ 0 & 1 & -4 \\ 1 & -1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

options: det of A if $\det A = 0 \rightarrow$ cols are dependent

or Gaussian elimination to a, b, c

or find A^{-1} if possible

$$\begin{bmatrix} 4 & -5 & 0 & 0 \\ 0 & 1 & -4 & 0 \\ 1 & -1 & -1 & 0 \end{bmatrix}$$

$$\rightarrow \cdots \rightarrow \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -4 & 0 \\ \boxed{0 \quad 0 \quad 0 \quad 0} \end{bmatrix}$$

↳ infinitely many ways to make $a\vec{u}+b\vec{v}+c\vec{w}=\vec{0}$

this means $a=b=c$ is **NOT** the only solution

therefore, $\vec{u}, \vec{v}, \vec{w}$ are NOT linearly independent

4.2 The Vector Space R^n and Subspaces

R^3 : space contains all 3D vectors

R^2 : " " " 2D vectors → xy-plane

R^1 : " " " 1D vectors (scalars) → line (real number line)

R^4 : " " " 4D vectors

R^5 : " " " 5D vectors

all of them behave <u>the same way</u> → they are in a space that contains 8 fundamental properties

1. $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ (commutativity)
2. $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$ (associative property)
3. there is a zero vector such that $\vec{u}+\vec{0} = \vec{u}$
4. $\vec{u}+(-\vec{u}) = (-\vec{u})+\vec{u} = \vec{0}$
5. $a(\vec{u}+\vec{v}) = a\vec{u}+a\vec{v}$
6. $(a+b)\vec{u} = a\vec{u}+b\vec{u}$

7. $a(b\vec{u}) = (ab)\vec{u}$

8. there is a number/vector "1" such that $1(\vec{u}) = \vec{u}$

in fact, these properties also hold for spaces such as

M_2 : space containing all 2x2 matrices

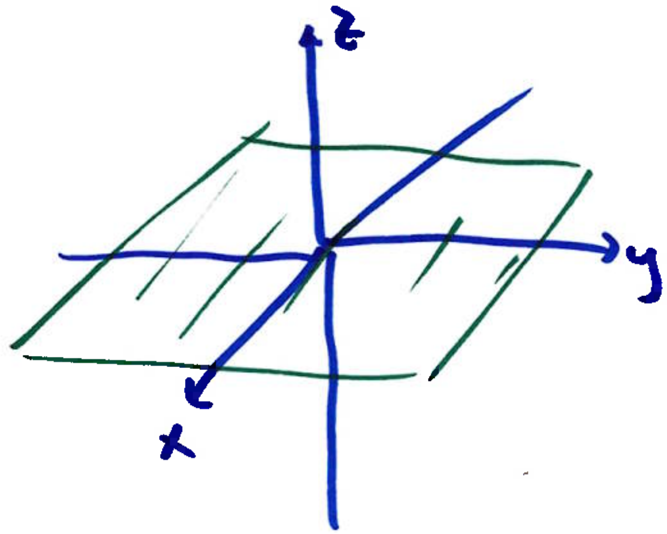
P_2 : all possible 2nd degree polynomials

All spaces that have these 8 properties are called

<u>vector spaces</u>

could be : number
vector
matrices
polynomials
etc

R^3: 3D space $\quad$ R^2: xy-plane



R^2 is contained in R^3

→ R^2 is a <u>subspace</u> of R^3

(R^1 is a subspace of R^2
R^{256} is a subspace of R^{257})

all vectors spaces (including subspaces) are

1. closed under addition
2. closed under scalar multiplication
3. containing zero "vector"

1. Sum of two R^2 vectors remain R^2 (two xy-plane vectors cannot be added to produce a vector w/ z-component)
2. Scalar multiple of a vector remains in the same space
3. must contain the origin

example R^2

two random vectors : $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ $\vec{v} = \begin{bmatrix} c \\ d \end{bmatrix}$

a, b, c, d are numbers

$\vec{u} + \vec{v} = \begin{bmatrix} a + c \\ b + d \end{bmatrix}$ a+c is another number

b+d " "

$= \begin{bmatrix} e \\ f \end{bmatrix}$ same kind as $\vec{u}, \vec{v}$ (R^2)

this shows that R^2 is closed under addition

to be continued...