4.2 Vector Space R^n and Subspaces (continued)

vector space: a "space" that has the 8 fundamental properties from last time

Subspace: a part of a vector space (which is a vector space itself)

→ satisfies ALL 8 properties

all can be summarized as

1. closure under addition
2. closure under scalar multiplication
3. contains the zero vector

example R^2 is a subspace of R^3

$\hookrightarrow$ xy-plane $\hookrightarrow$ 3D coord. system

closure under addition means sum of two vectors from this space remains in the space

$\rightarrow$ here, two vectors from xy-plane whose sum remains in the xy-plane

we show it this way:

let $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} c \\ d \end{bmatrix}$ be two R^2 vectors a, b, c, d reals

$\vec{u} + \vec{v} = \begin{bmatrix} a+c \\ b+d \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$ e, f are ~~agin,~~ again, reals (sum of reals is real)

this shows closure under addition

closure under multiplication:

$\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ $\quad k\vec{u} = \begin{bmatrix} ka \\ kb \end{bmatrix}$ $\quad$ k, a, b are reals

$\rightarrow$ ka, kb also reals

clearly still in R^2

(this shows closure)

last condition: $\vec{0}$ is in the space

R^2: $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ $\quad$ a, b reals

so, $\vec{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is also contained in this space

(origin) is part of xy-plane

ALL 3 conditions are true (all 8 fundamental properties hold) so R^2 is a subspace (and vector space)

Example

W is a part of R^3 such that $y = -1$

Is W a subspace?

a part of a vector space is NOT automatically
a subspace!

— it must satisfy the three conditions

W contains vectors that look like

$$\begin{bmatrix} x \\ -1 \\ z \end{bmatrix} \qquad x, z \text{ are reals}$$

add two of them: $\begin{bmatrix} x_1 \\ -1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ -1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ -2 \\ z_1 + z_2 \end{bmatrix}$

Since middle component is
NOT -1 so sum of two is
NOT in W anymore

so, NOT closed under addition

So W is NOT a subspace

a subspace we will frequently see is the <u>null space</u>

$\rightarrow$ contains ALL solutions to $A\vec{x} = \vec{0}$

(all $\vec{x}$ that get sent to $\vec{0}$)

<u>example</u>

$$\begin{aligned} x_1 - 4x_2 - 3x_3 - 7x_4 &= 0 \\ 2x_1 - x_2 + x_3 + 7x_4 &= 0 \\ x_1 + 2x_2 + 3x_3 + 11x_4 &= 0 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix}}_{A} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & -3 & -7 & 0 \\ 2 & -1 & 1 & 7 & 0 \\ 1 & 2 & 3 & 11 & 0 \end{bmatrix}$$

$$\rightarrow \cdots \rightarrow \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \begin{bmatrix} \boxed{1} & 0 & 1 & 5 & 0 \\ 0 & \boxed{1} & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

0 zero → free variables : $x_3 = r$

$x_4 = t$

row 2 : ... → $x_2 = -r - 3t$

row 1 : ... → $x_1 = -r - 5t$

all $\vec{x}$ such that $A\vec{x} = \vec{0}$ look like

$$\begin{bmatrix} -r - 5t \\ -r - 3t \\ r \\ t \end{bmatrix} = \underbrace{r \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{\text{linear combo of } \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}}$$

these are the "contents" of null space of A

4.3 Linear Combos and Linear Independence

given $\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_n$

linearly indp if and only if $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0}$

means $c_1 = c_2 = \dots = c_n = 0$

if we rewrite it this way:

$\vec{v}$ as columns

$$\underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

if $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly indp, then

the only content of null space of A is $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0}$

if A is square, then $\det A \neq 0$ (A^{-1} exists)

but what if A is not square?

example $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\vec{v_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\vec{v_4} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

are they linearly indep ?

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

we want $c_1 = c_2 = c_3 = c_4 = 0$ being the unique solution

↳ cannot have row of all zeros or a variable column w/o pivot } free variables

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 3 & 0 \end{bmatrix}$$

$$\to \cdots \to \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} \boxed{1} & 0 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & 2 & 0 \\ 0 & 0 & \boxed{1} & 2 & 0 \end{bmatrix} \end{array}$$

3 pivots, 4 variables → one free, solution NOT unique → not linearly indep

if vectors have m components ($\mathbb{R}^m$ vectors)
and there are more than m of them in a set
then at least one column will NOT have a pivot
→ as a set, there are NOT linearly indp.

but if there m or fewer, then we can't
immediately tell → reduce and count pivots

<u>example</u> $\vec{v}_1 = \begin{bmatrix} -1 \\ -17 \\ -3 \\ 9 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 14 \\ 7 \\ 2 \\ -2 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 15 \\ 5 \\ 1 \\ -2 \end{bmatrix}$

three $\mathbb{R}^4$ vectors, may or may not be indp.

$$\begin{bmatrix} -1 & 14 & 15 \\ -17 & 7 & 5 \\ -3 & 2 & 1 \\ 9 & -2 & -2 \end{bmatrix}$$

$$\to \cdots \to \begin{bmatrix} \boxed{-1} & 14 & 15 \\ 0 & \boxed{-231} & -250 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

3 pivots, 3 vectors
→ no free variables
despite
zero row

span : the things we can make w/ given vectors

example : $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

using linear combos of $\vec{i}$, $\vec{j}$, we can make ~~every p~~ every possible R^2 vector

so we say $\vec{i}$ and $\vec{j}$ span R^2

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = R^2$$

likewise, $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = R^3$

but $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ along do NOT span R^3

does $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ span R^3 ?

yes, there is a redundant one → not linearly indp

note we need a minimum of n vectors to span R^n