5.3 Homogeneous Eqs. with Constant Coefficients

$y'' + ay' + by = 0$ a, b constants

as long as the coeffs are constant

the solutions are always $y = e^{rx}$.

where r is the solution to the charactoristic eq.

n of these for n^{th}-order.

for example, $2y'' - 3y' = 0$

characteristic eq: $2r^2 - 3r = 0$

$r(2r-3) = 0$

$r = 0, \; r = 3/2$

solutions: $y_1 = e^{0x} = 1$

$y_2 = e^{3/2 x}$

general solution: $y = c_1 \cdot 1 + c_2 \cdot e^{3/2 x}$

when we solve the char. eq : solutions can be

real and distinct (last example)

real and repeated

complex

repeated: $y'' - 2y' + y = 0$

$r^2 - 2r + 1 = 0$

$(r-1)(r-1) = 0 \qquad r = 1, 1$

solution: $y = e^{rx}$ $\qquad y_1 = e^x$

$y_2 = x e^x$

extra x : y_2 is linearly indp from y_1

why? recall the method of reduction of order

$y_2 = v(x) y_1$

sub into eq. ... $v(x) = x$

higher-order: $y^{(4)} - 8y''' + 16y'' = 0$

$$r^4 - 8r^3 + 16r^2 = 0$$

$$r^2(r^2 - 8r + 16) = 0$$

$$(r)(r)(r-4)(r-4) = 0$$

$$r = \underbrace{0, 0}_{\text{repeated}}, \underbrace{4, 4}_{\text{repeated}}$$

$$y_1 = e^{0x} = 1$$

$$y_2 = x e^{0x} = x$$

$$y_3 = e^{4x}$$

$$y_4 = x e^{4x}$$

general solution: $y = c_1 + c_2 x + c_3 e^{4x} + c_4 x e^{4x}$

Cubic char. eqs are tricky to factor

<u>example</u> $y''' + 5y'' - 100y' - 500y = 0$

char. eq: $r^3 + 5r^2 - 100r - 500 = 0$

if the coeffs have same pattern:

$1, \; 5, \; -100, \; -500$

mult. 5 mult. 5

$$r^3 + 5r^2 - 100r - 500 = 0$$

$$r^2(r+5) - 100(r+5) = 0$$

$$(r+5)(r^2 - 100) = 0$$

$$(r+5)(r-10)(r+10) = 0 \qquad r = -5, 10, -10$$

$$y = c_1 e^{-5x} + c_2 e^{10x} + c_3 e^{-10x}$$

<u>example</u> $3y''' + 4y'' - 5y' - 2y = 0$

char. eq. $3r^3 + 4r^2 - 5r - 2 = 0$

no common multiple pattern

one root can be found this way:

last number: -2

factors: $-2 \begin{cases} -2 \\ +1 \end{cases}$ or $-2 \begin{cases} 2 \\ -1 \end{cases}$

at least one is ~~the~~ a root

for example, $r=1$ works

$3r^3 + 4r^2 - 5r - 2 = 0$

$\hookrightarrow \underbrace{(r-1)}_{\text{because } r=1} \underbrace{(ar^2 + br + c)}_{?} = 0$

long division:

$3r^3+4r^2-5r-2 = (r-1)(ar^2+br+c)$

$$ar^2+br+c = \frac{3r^3+4r^2-5r-2}{r-1}$$

$$\begin{array}{r}
3r^2+7r+2 \\
r-1 \enclose{longdiv}{3r^3+4r^2-5r-2} \\
-(3r^3-3r^2) \\
\hline
7r^2-5r-2 \\
-(7r^2-7r) \\
\hline
2r-2 \\
-(2r-2) \\
\hline
0
\end{array}$$

the quadratic we want

$(r-1)(3r^2+7r+2)=0$

$(r-1)(3r+1)(r+2)=0 \qquad r=1,\ -1/3,\ -2$

$y=c_1e^{x}+c_2e^{-1/3x}+c_3e^{-2x}$

Complex roots.

$y'' + y = 0$ char. eq. $r^2+1=0$ $r^2=-1$

$r = \pm i$ where $i^2=-1$

Solution: $y_1 = e^{ix}$

$y_2 = e^{-ix}$

$y = c_1 e^{ix} + c_2 e^{-ix} \rightarrow ?$

real-valued

solution of $y''+y=0 \rightarrow y'' = -y$

by reasoning: y is negative of its 2nd-deriv.

$y = \cos x$ and $\sin x$

$y' = -\sin x$ $\cos x$

$y'' = -\cos x$ $-\sin x$

e^{ix} must be related to $\cos x$ and $\sin x$ somehow.

remember $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$i = i \quad i^5 = i$

$i^2 = -1 \quad i^6 = -1$

$i^3 = -i \quad i^7 = -i$

$i^4 = 1 \quad i^8 = 1$

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} + \cdots$$

$$= \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)}_{\cos x} + i\underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)}_{\sin x}$$

$$\boxed{e^{ix} = \cos x + i\sin x}$$

Euler's formula

$$e^{i(ax)} = \cos(ax) + i\sin(ax)$$

<u>example</u> $y'' + 100y = 0$

$r^2 + 100 = 0 \qquad r^2 = -100 \qquad \underbrace{r = 10i, -10i}$

complex roots are ALWAYS in conjugate <u>pairs</u>

Solutions: $e^{10ix} = e^{i(10x)} = \cos(10x) + i\sin(10x)$

$e^{-10ix} = e^{i(-10x)} = \cos(-10x) + i\sin(-10x)$

$= \cos(10x) - i\sin(10x)$

two solutions also in conjugate pairs

we want solutions to be real-valued so we use the real and imaginary parts of either solutions as y_1 and y_2

$y_1 = \cos(10x)$ real part

$y_2 = \sin(10x)$ imag part

general solution: $\boxed{y = C_1 \cos(10x) + C_2 \sin(10x)}$

<u>example</u> $y'' - 6y' + 25y = 0$

$$r^2 - 6r + 25 = 0$$

$$r = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(25)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{-64}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i$$

form one solution: $e^{(3+4i)x}$

$$= e^{3x} e^{4ix} = e^{3x}\left(\cos(4x) + i\sin(4x)\right)$$

$$= \underbrace{e^{3x}\cos(4x)}_{\text{real}} + i\,\underbrace{e^{3x}\sin(4x)}_{\text{imag.}}$$

$$\boxed{y = c_1 e^{3x}\cos(4x) + c_2 e^{3x}\sin(4x)}$$

example $6y^{(4)} + 11y'' + 4y = 0$

$6r^4 + 11r^2 + 4 = 0$

4 roots: all real

two pairs of complex

two real and two complex

$6u^2 + 11u + 4 = 0 \qquad u = r^2$

$$u = \frac{-11 \pm \sqrt{121 - 96}}{2(6)} = -\frac{4}{3}, \ -\frac{1}{2}$$

$r^2 = -\frac{4}{3} \qquad r^2 = -\frac{1}{2}$

$$r = \underbrace{\frac{2}{\sqrt{3}}i, \ -\frac{2}{\sqrt{3}}i}, \ \underbrace{\frac{1}{\sqrt{2}}i, \ -\frac{1}{\sqrt{2}}i}$$

$y_1 = \cos\left(\frac{2}{\sqrt{3}}x\right) \qquad y_3 = \cos\left(\frac{1}{\sqrt{2}}x\right)$

$y_2 = \sin\left(\frac{2}{\sqrt{3}}x\right) \qquad y_4 = \sin\left(\frac{1}{\sqrt{2}}x\right)$

$$y = c_1 \cos\left(\frac{2}{\sqrt{3}}x\right) + c_2 \sin\left(\frac{2}{\sqrt{3}}x\right) + c_3 \cos\left(\frac{1}{\sqrt{2}}x\right) + c_4 \sin\left(\frac{1}{\sqrt{2}}x\right)$$