7.1 First-order systems of Diff. Eqs

system of eqs.

system of two 1st-order : $x'(t) = f(x, y, t)$

$y'(t) = g(x, y, t)$

x, y : dependent variable

t : indp variable

for example, $x' = y$

$y' = -x$

notice to solve for x, we need to know y

(and vice versa)

→ these eqs. are <u>coupled</u>

they need to be solved simultaneously

if the eqs are simple like these, we can solve the system by converting a system of two 1st-order into one 2nd-order eq.

$$\begin{cases} x' = y \\ y' = -x \end{cases}$$

differentiate : $x'' = y'$

$x'' = -x \rightarrow \boxed{x'' + x = 0}$

we can solve it : $r^2 + 1 = 0$

$r = \pm i$

$$\boxed{x(t) = C_1 \cos(t) + C_2 \sin(t)}$$

from $x' = y$, we get $\boxed{y(t) = -C_1 \sin(t) + C_2 \cos(t)}$

we can actually go the other way: one 2nd-order → sys of two 1st-order

$$\boxed{\text{System of } n \text{ 1st-order} \longleftrightarrow \text{one } n^{th}\text{-order}}$$

n^{th}-order → system of n 1st-order is important in solving eqs. numerically (e.g. w/ Matlab)

<u>example</u> 2nd-order → system of 1st-order

$$x'' + 3x' + 7x = t^2$$

define two variables to represent

$$\underbrace{x \text{ and } x'}$$

define new variables for derivatives of x below the highest one

let $z_1 = x$

$z_2 = x'$

notice $\boxed{z_1' = z_2}$ 1st eq. in the system (consequence of definition)

the second eq. is the original one using new variables

$$x'' + 3x' + 7x = t^2$$

$z_1 = x$

$z_2 = x'$

$$\underbrace{x''}_{z_2'} = -3\underbrace{x'}_{z_2} - 7\underbrace{x}_{z_1} + t^2$$

$\boxed{z_2' = -3z_2 - 7z_1 + t^2}$ 2nd eq. in the system

<u>example</u> $x^{(4)} + 6x''' - 3x'' + x' + 10x = \cos(3t)$

define variables to represent derivs. of x below the highest one: x''', x'', x', x

$z_1 = x$

$z_2 = x'$

$z_3 = x''$

$z_4 = x'''$

$$\left.\boxed{\begin{aligned} z_1' &= z_2 \\ z_2' &= z_3 \\ z_3' &= z_4 \end{aligned}}\right\} \text{ consequence of definition of } z_n$$

last one: original eq. in new variables

$$x^{(4)} + 6x''' - 3x'' + x' + 10x = \cos(3t)$$

$$x^{(4)} = -6x''' + 3x'' - x' - 10x + \cos(3t)$$

$$\boxed{z_4' = -6z_4 + 3z_3 - z_2 - 10z_1 + \cos(3t)}$$

n^{th}-order $\rightarrow$ n 1st-order

sys. of two 2nd-order $\rightarrow$ sys. of four 1st-order example

$$x'' - 5x' - 4x + 6y = 0$$
$$y'' + 6y' + 5x + 5y = 0$$

define variables to represent : $x,\ x',\ y,\ y'$

$$z_1 = x$$
$$z_2 = x'$$
$$z_3 = y$$
$$z_4 = y'$$

$$x'' = 5x' + 4x - 6y$$

$$z_1' = z_2$$
$$z_2' = 5z_2 + 4z_1 - 6z_3$$
$$z_3' = z_4$$
$$z_4' = -6z_4 - 5z_1 - 5z_3$$

rewrite as matrix equation:

$$\begin{bmatrix} z_1' \\ z_2' \\ z_3' \\ z_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4 & 5 & -6 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 0 & -5 & -6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

this is a matrix eq. in the form of

$$\vec{x}\,' = \underbrace{A}_{\text{coefficient matrix}} \vec{x} \quad \text{where} \quad \vec{x} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

7.2 Matrices and Linear Systems

- some basic properties of matrix differential eq.

more general system: $t^2 x'' + t x' + (t^2-1)x = e^t$

$$x'' + \frac{1}{t}x' + \frac{t^2-1}{t^2}x = \frac{e^t}{t^2}$$

define $z_1 = x$

$z_2 = x'$

$z_1' = z_2$

$$z_2' = -\frac{1}{t}z_2 - \frac{t^2-1}{t^2}z_1 + \frac{e^t}{t^2}$$

$$\underbrace{\begin{bmatrix} z_1' \\ z_2' \end{bmatrix}}_{\vec{x}'} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{t^2-1}{t^2} & -\frac{1}{t} \end{bmatrix}}_{P(t)} \underbrace{\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} 0 \\ \frac{e^t}{t^2} \end{bmatrix}}_{\vec{f}(t)}$$

in the form of $\vec{x}' = P(t)\vec{x} + \vec{f}(t)$

if $\vec{f}(t) = \vec{0}$, the eq. is <u>homogeneous</u> $(\vec{x}' = P(t)\vec{x})$

if $\vec{f}(t) \neq \vec{0}$, " " <u>nonhomogeneous</u>

the general solution of $\vec{x}' = P\vec{x} + \vec{f}$ is

$$\vec{X}(t) = \vec{X}_c + \vec{x}_p$$

complementary solution (no $\vec{f}$)

particular solution (caused by $\vec{f}$)

if P is $n \times n$, then there are n solutions

$$\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$$

(because $n \times n \to n^{th}$-order $\to$ n solutions) these n solutions are linearly independent

→ ~~these~~ ~~as~~ their Wronskian is not zero on some interval of t

$$W(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n) = \begin{vmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \cdots & \vec{x}_n \end{vmatrix}$$

each solution as columns

if $W \neq 0$ on an interval, then the solutions are linearly indp on that interval

$$\vec{x}_c = c_1 \vec{x}_1 + c_2 \vec{x}_2 + c_3 \vec{x}_3 + \cdots + c_n \vec{x}_n$$

<u>example</u> Verify $\vec{x}_1 = e^{2t}\begin{bmatrix}1\\1\end{bmatrix}$ and $\vec{x}_2 = e^{-2t}\begin{bmatrix}1\\5\end{bmatrix}$

are solutions of $\vec{x}\,' = \begin{bmatrix}3 & -1\\5 & -3\end{bmatrix}\vec{x}$

and show that they are linearly indp.

Solution: satisfies $\vec{x}\,' = A\vec{x}$

$$\left.\begin{aligned}\vec{x}_1 &= e^{2t}\begin{bmatrix}1\\1\end{bmatrix}\\ \vec{x}_1\,' &= 2e^{2t}\begin{bmatrix}1\\1\end{bmatrix}\end{aligned}\right\} \text{sub into } \vec{x}\,' = \begin{bmatrix}3 & -1\\5 & -3\end{bmatrix}\vec{x}$$

$$2e^{2t}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3 & -1\\5 & -3\end{bmatrix}e^{2t}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= e^{2t}\begin{bmatrix}3 & -1\\5 & -3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= e^{2t}\begin{bmatrix}2\\2\end{bmatrix} \quad \text{(match)}$$

so, $\vec{x}_1$ is a solution (easy to show $\vec{x}_2$ is also)

indp? $\vec{x}_1 = e^{2t}\begin{bmatrix}1\\1\end{bmatrix}$ $\vec{x}_2 = e^{-2t}\begin{bmatrix}1\\5\end{bmatrix}$

$$W = \begin{vmatrix} e^{2t} & e^{-2t} \\ e^{2t} & 5e^{-2t} \end{vmatrix} = 5e^{-2t}e^{2t} - e^{2t}e^{-2t}$$

$= 4 \neq 0$ on $-\infty < t < \infty$

so, $\vec{x}_1$, $\vec{x}_2$ are indp on $-\infty < t < \infty$

Solution: $\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$

$$= c_1 e^{2t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{-2t}\begin{bmatrix}1\\5\end{bmatrix}$$

where did those solutions come from?

$$\vec{x}\,' = \underbrace{\begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}}_{A} \vec{x} \qquad \vec{x}_1 = e^{\boxed{2}t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{x}_2 = e^{\boxed{-2}t}\begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

eigenvalue of A: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 3-\lambda & -1 \\ 5 & -3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-3-\lambda) + 5 = 0$$

$$\lambda^2 - 4 = 0 \qquad \lambda = \boxed{2},\ \boxed{-2}$$

eigenvector of $\lambda = 2$: $(A - \lambda I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 5 & -5 & 0 \end{array}\right]$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_2 = r$

$x_1 = x_2 = r$

$$\vec{v} = \begin{bmatrix} r \\ r \end{bmatrix} = r \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

choose $r = 1$

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$