7.4 Solution Curves of Linear Systems

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x} \qquad \lambda = 1, -1$$

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

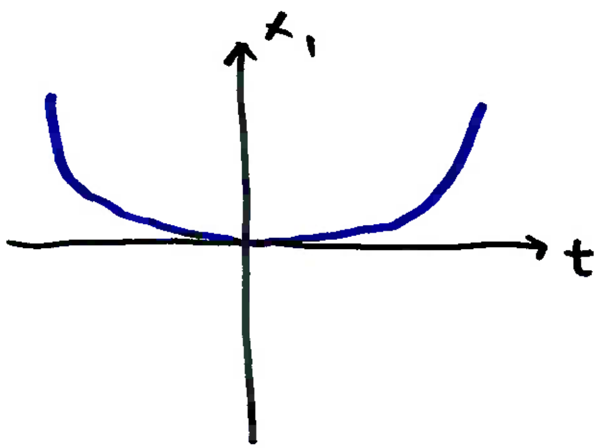
solution: $\vec{x}(t) = c_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

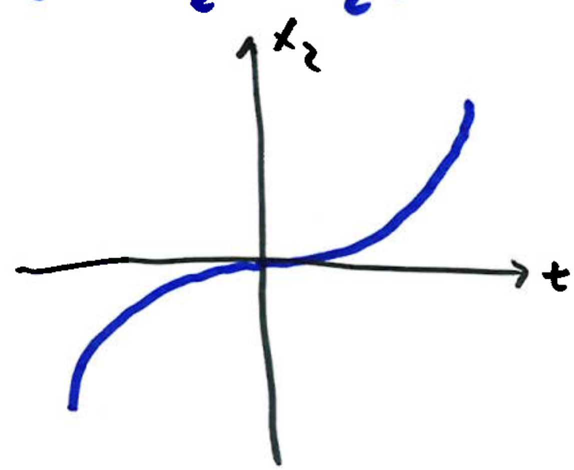
$$x_1(t) = c_1 e^{t} + c_2 e^{-t}$$

$$x_2(t) = c_1 e^{t} - c_2 e^{-t}$$

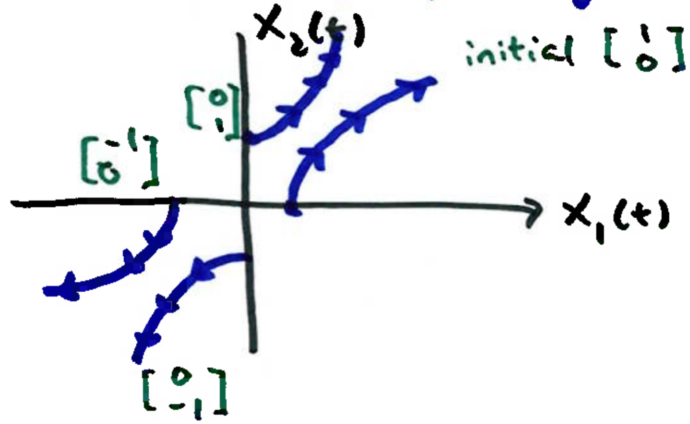
if $\vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \quad c_1 = \frac{1}{2} \quad c_2 = \frac{1}{2}$

$$x_1(t) = \frac{1}{2} e^{t} + \frac{1}{2} e^{-t} \qquad x_2(t) = \frac{1}{2} e^{t} - \frac{1}{2} e^{-t}$$





if we graph $x_1(t)$ vs $x_2(t)$, we get a phase curve



we need to do this for each initial condition $\vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$

there is a more efficient way to complete the <u>phase portrait</u>

revisit $\vec{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x}$ $\quad \lambda = 1, -1$

$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

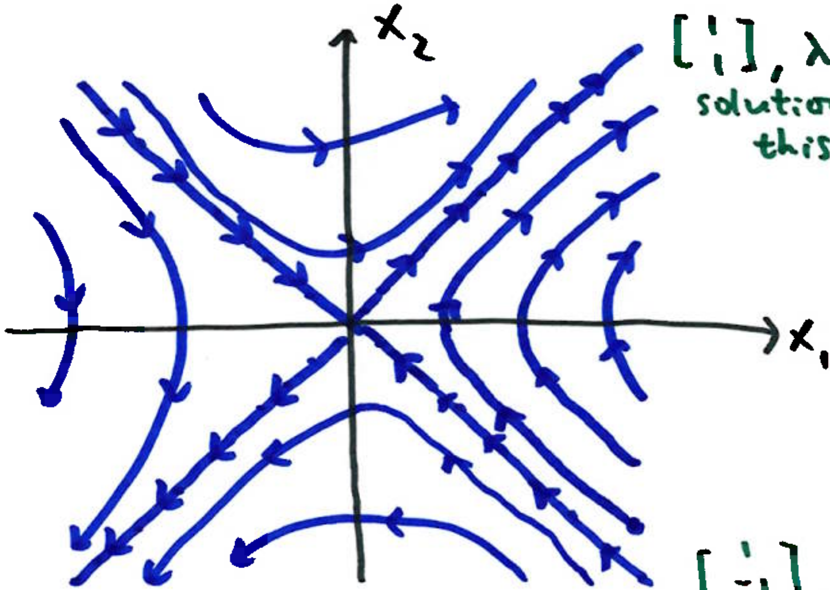
$$\vec{x}(t) = c_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

as $t \to \infty$, $e^{-t} \to 0$, so $\vec{x}(t) \approx c_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad x_1 \approx x_2$

as $t \to -\infty$, $e^{t} \to 0$, so $\vec{x}(t) \approx c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \to$ $\quad x_1 \approx -x_2$

the vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (eigenvectors) are important

all nearby solution curves want to follow $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



if initial condition is on $\begin{bmatrix}1\\1\end{bmatrix}$, for example

$\vec{x}(0) = \begin{bmatrix}1\\1\end{bmatrix}$

$\cdots \to \vec{x}(t) = \begin{bmatrix}e^t\\e^t\end{bmatrix}$

both $x_1, x_2 \to \infty$ as $t \to \infty$

for init. cond on $\begin{bmatrix}1\\-1\end{bmatrix}$, $\vec{x}(t) = \begin{bmatrix}e^{-t}\\-e^{-t}\end{bmatrix}$

all solutions ~~on~~ w/ init. cond. NOT on the lines use them as asymptotes

the eigenvalues for straight line solutions

if $\lambda > 0 \to$ flow away from origin

if $\lambda < 0 \to$ flow toward origin

nearby solutions use them as asymptotes

here, the origin is a saddle point (solutions sometimes approach and sometimes go away from origin) $\to$ $\boxed{\text{eigenvalues have opposite signs}}$

$\vec{x}' = \begin{bmatrix} 6 & 5 \\ -3 & -2 \end{bmatrix} \vec{x}$

$\lambda = 3,\ 1$

$\vec{v} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\underbrace{x(t) = c_1 e^{3t} \begin{bmatrix} -5 \\ 3 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

notice the origin $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

is at $t = -\infty$

as t increases from $t = -\infty$

$e^{3t} \ll e^{t}$

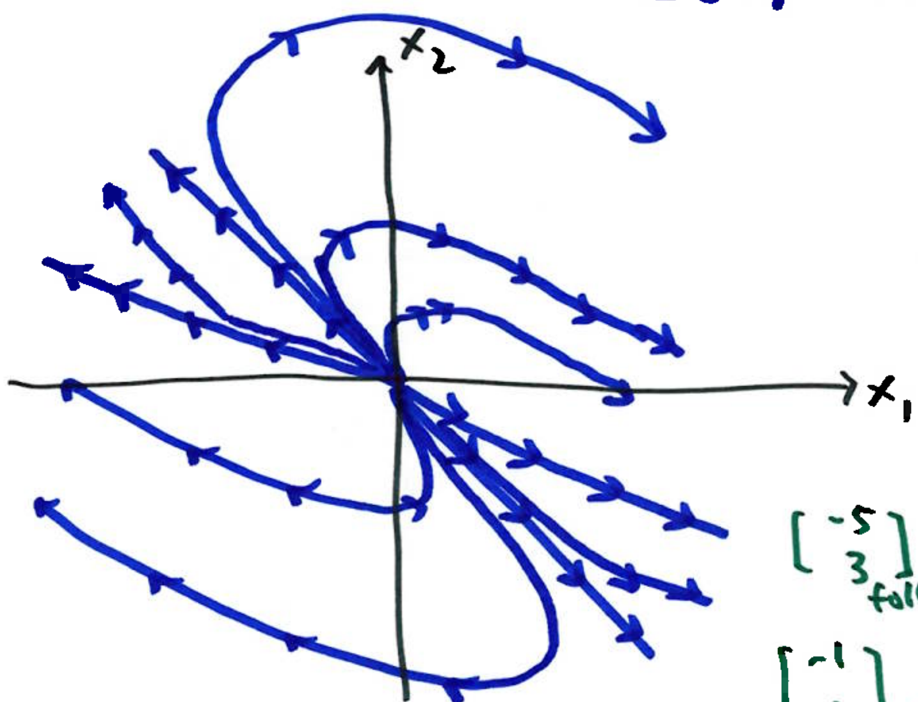
solutions follow $e^{t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

then as t keeps increasing

$e^{3t} \gg e^{t}$

solutions follow

$e^{3t} \begin{bmatrix} -5 \\ 3 \end{bmatrix}$



$\begin{bmatrix} -5 \\ 3 \end{bmatrix}$, $\lambda > 0$

follow this later

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\lambda > 0$

follow this first

the origin here is called an improper nodal source

$\boxed{\lambda\text{'s are distinct}}$

solutions near origin follow asymptote and are ~~di~~ indistinguishable

solutions come out of origin

if both λ's are negative, the origin is an improper nodal sink

$\boxed{\text{λ's are distinct}}$

solutions sink into origin

$$\vec{x}\,' = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \vec{x}$$

$\lambda = i\ ,\ -i$

$\vec{v} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}, \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$

$$\vec{x}(t) = c_1 \begin{bmatrix} \sin t + \cos t \\ 2\sin t \end{bmatrix} + c_2 \begin{bmatrix} -\sin t \\ \cos t - \sin t \end{bmatrix}$$

to analyze shape, let's pick a convenient init. cond.: $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

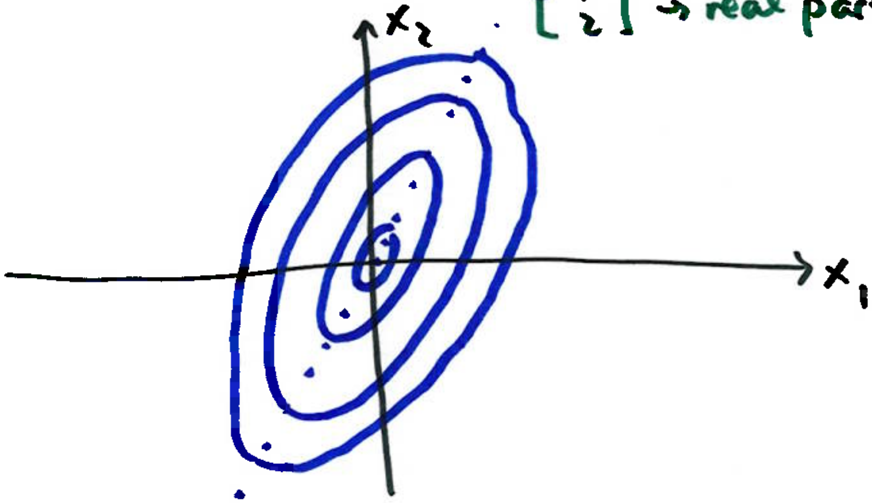
$\cdots \rightarrow c_1 = 1,\ c_2 = 0 \rightarrow \vec{x}(t) = \begin{bmatrix} \sin t + \cos t \\ 2\sin t \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$x_1 = \sin t + \cos t \qquad x_2 = 2\sin t$

notice $\Big(\underbrace{x_1 - \frac{x_2}{2}}_{\cos t}\Big)^2 + \Big(\underbrace{\frac{x_2}{2}}_{\sin t}\Big)^2 = 1 \longrightarrow$ ellipse w/ center at origin

major axis (longer axis) is along $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow$ real part of eigenvector

init. cond. affects ellipse size



$\begin{bmatrix}1\\2\end{bmatrix} \to$ real part of e-vector

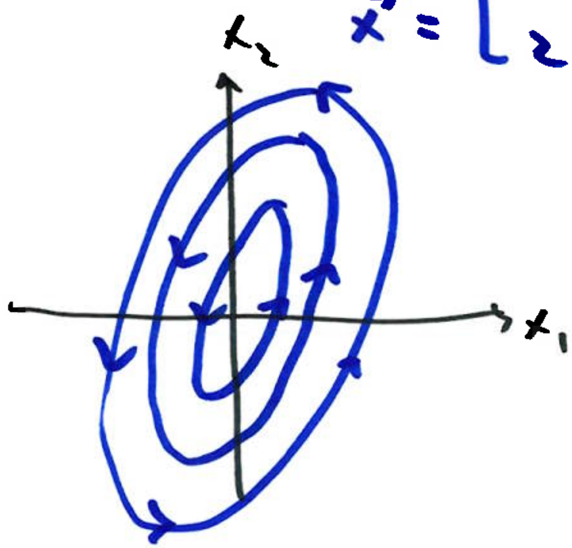
the solutions can go clockwise or CCW

to see direction: $\vec{x}\,' = \begin{bmatrix}1 & -1\\2 & -1\end{bmatrix}\vec{x}$

pick a convenient $\vec{x}$: $\vec{x} = \begin{bmatrix}1\\0\end{bmatrix}$

$\vec{x}\,' = \begin{bmatrix}1 & -1\\2 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$ one right, two up

CCW



this is w/ purely imaginary eigenvalues

origin is called a center

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}$$

$\lambda = 1-i,\ 1+i$

$\vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$$\vec{x}(t) = c_1 e^t \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

pick a convenient init. cond. to analyze shape: $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

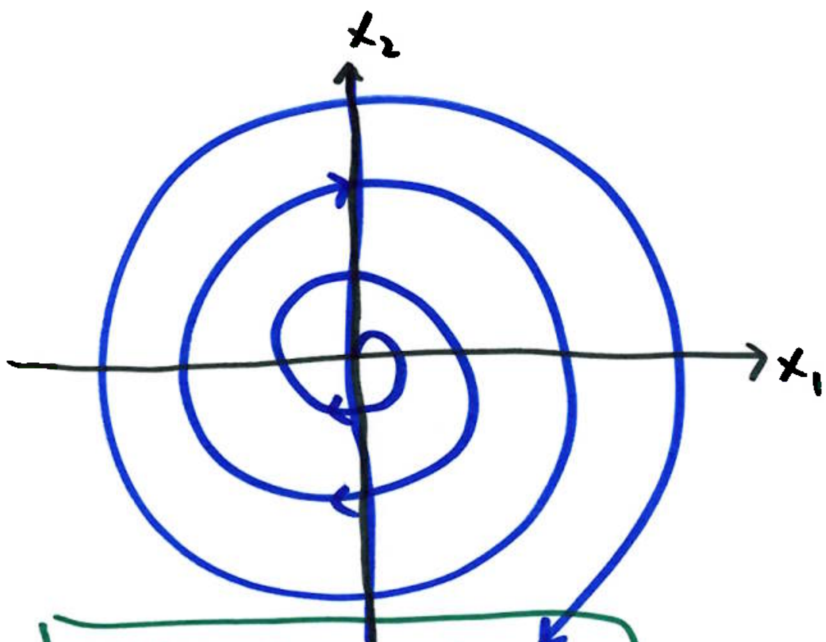
$\rightarrow c_1 = 1,\ c_2 = 0$

$$\vec{x}(t) = e^t \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$$

$x_1 = e^t \cos t$

$x_2 = -e^t \sin t$

$x_1^2 + x_2^2 = e^{2t}$ $\rightarrow$ circle w/ increasing radius $\rightarrow$ spiral



$\boxed{\text{Spiral source (if out)} \quad \text{spiral sink (if in)}}$

cw or ccw

pick $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ cw

spiral <u>out</u> if real part of λ is positive

spiral <u>in</u> if real part of λ is negative

$$\vec{x}' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \vec{x}$$

$\lambda = -1, -1$

$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ } complete (enough e-vectors)

$$\vec{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

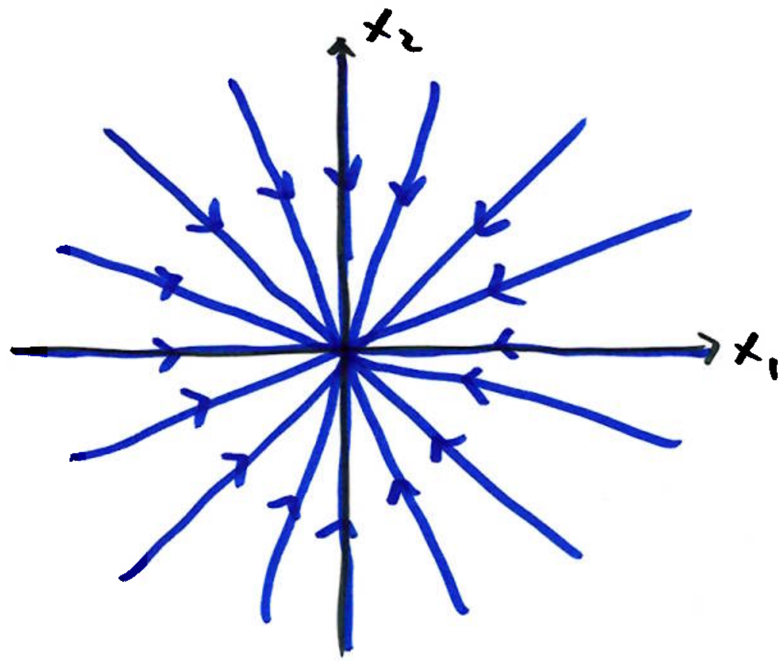
$$x_1 = c_1 e^{-t} \qquad x_2 = c_2 e^{-t}$$

$$\frac{x_2}{x_1} = \frac{c_2}{c_1} = m \rightarrow x_2 = m x_1$$

lines w/ slope m

into origin if $\lambda < 0$

out of origin if $\lambda > 0$



proper nodal sink

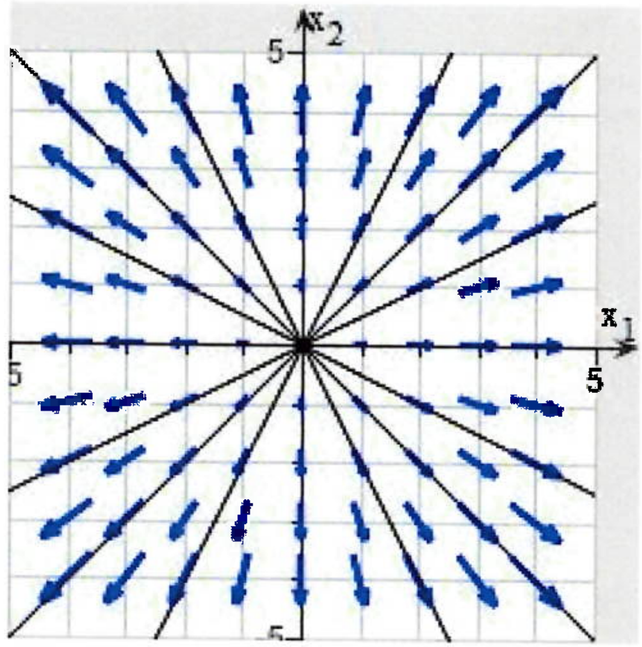
$\downarrow$

distinguishable near origin (no asymptote)

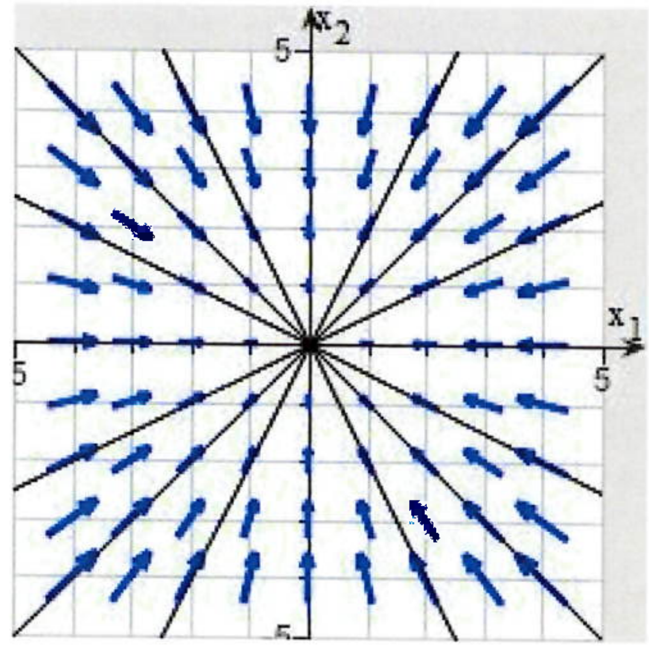
proper is rare

repeated eigenvalues and enough eigenvectors

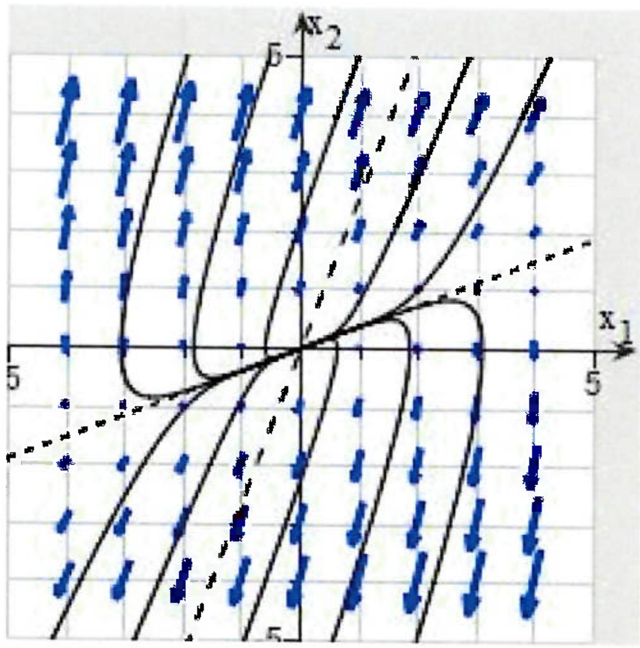
Gallery of Typical Phase Portraits for the System x'=Ax: Nodes

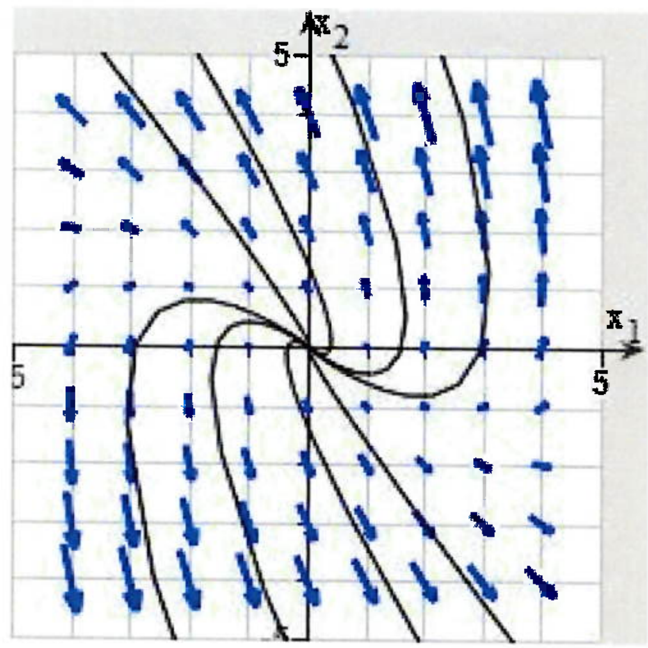


Proper Nodal Source: A repeated positive real eigenvalue with two linearly independent eigenvectors.



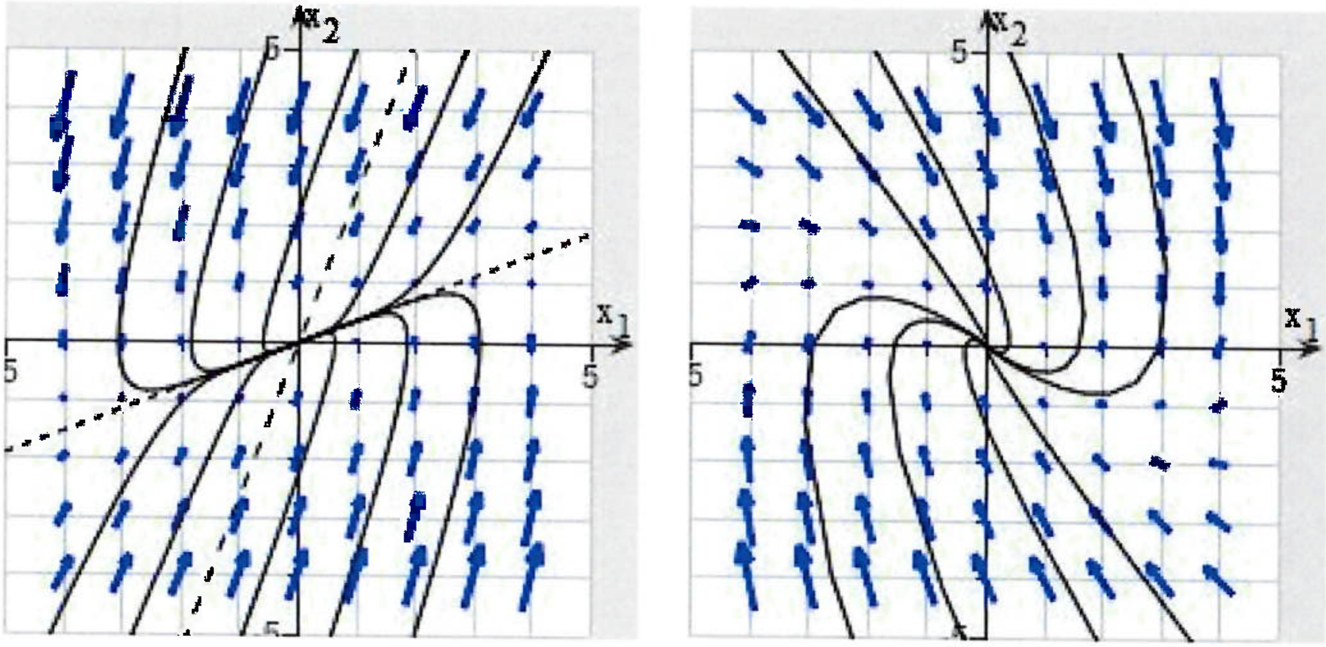
Proper Nodal Sink: A repeated negative real eigenvalue with two linearly independent eigenvectors.



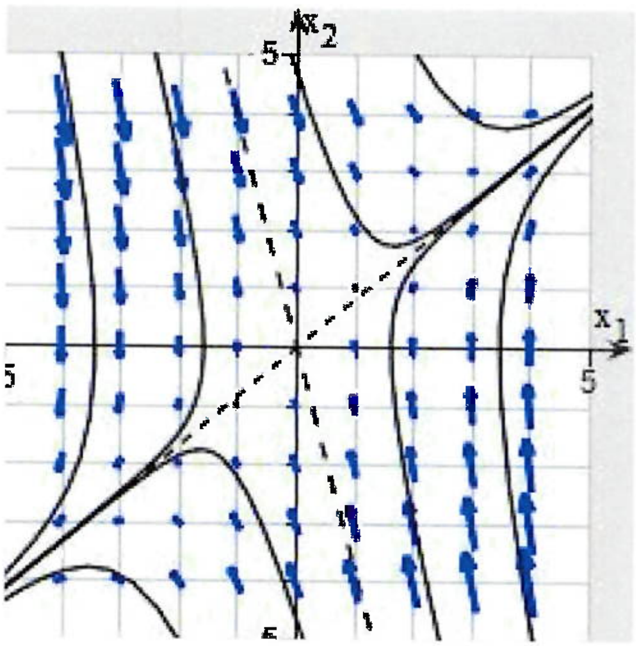


Improper Nodal Source: Distinct positive real eigenvalues (left) or a repeated positive real eigenvalue without two linearly independent eigenvectors (right).

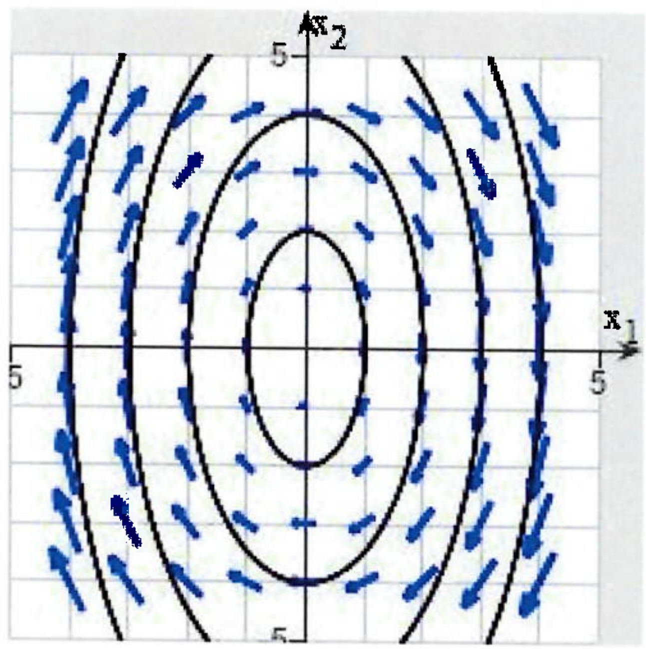
Gallery of Typical Phase Portraits for the System x'=Ax: Nodes



Improper Nodal Sink: Distinct negative real eigenvalues (left) or a repeated negative real eigenvalue without two linearly independent eigenvectors (right).

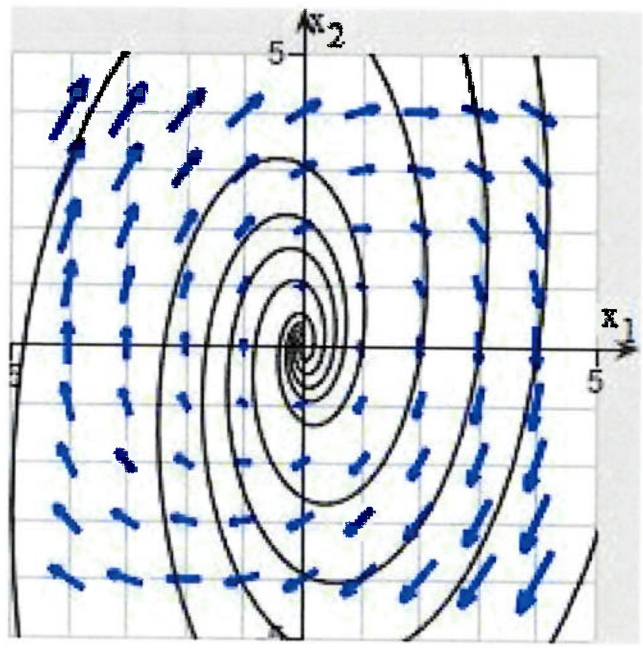


Saddle Point: Real eigenvalues of opposite sign.

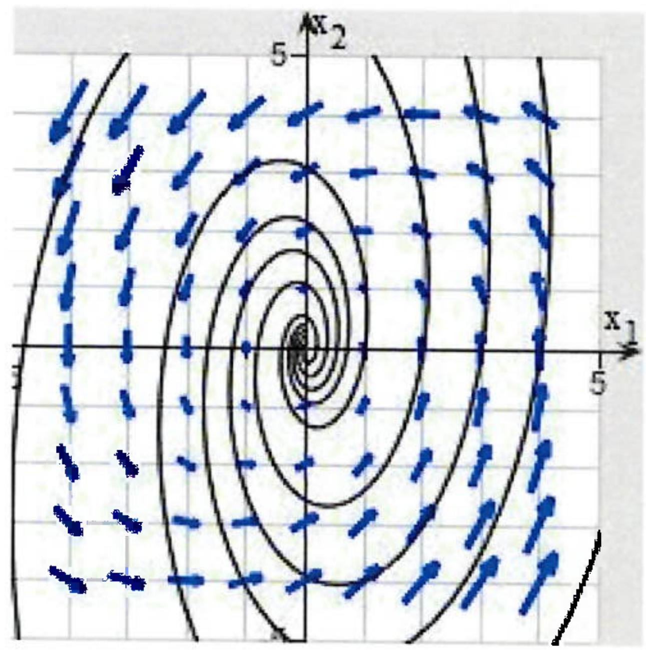


Center: Pure imaginary eigenvalues.

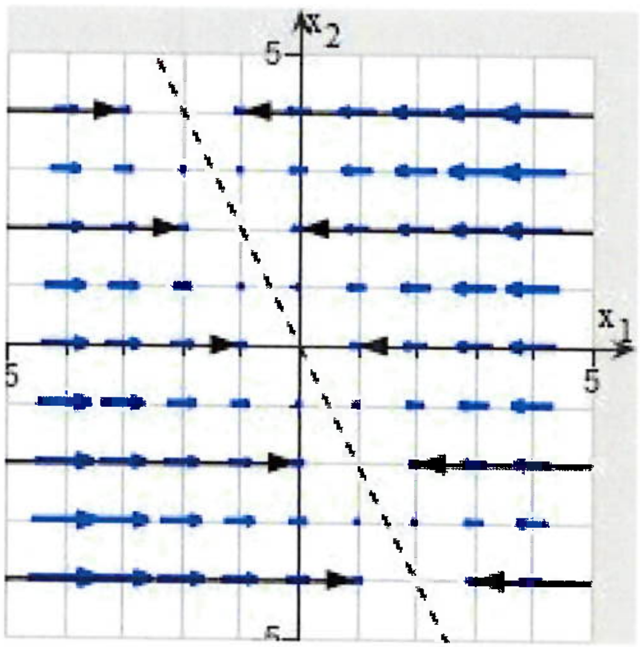
Gallery of Typical Phase Portraits for the System x'=Ax: Nodes



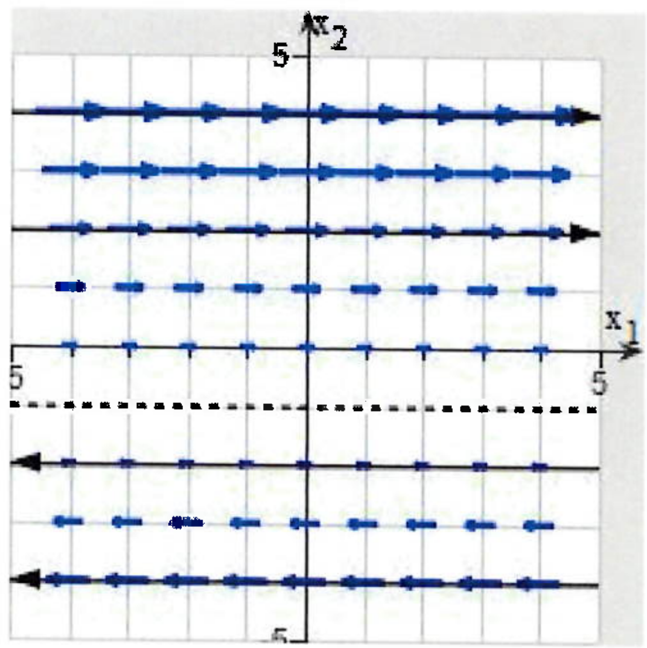
Spiral Source: Complex conjugate eigenvalues with positive real part.



Spiral Sink: Complex conjugate eigenvalues with negative real part.



Parallel Lines: One zero and one negative real eigenvalue. (If the nonzero eigenvalue is positive, then the trajectories flow away from the dotted line.)



Parallel Lines: A repeated zero eigenvalue without two linearly independent eigenvectors.