1.4 Separable Differential Eqs.

recall $\frac{dy}{dx} = 2x$ solution is easy because only x on right side

$y = \int 2x\,dx = x^2 + C$ easy because x and y are not together

but $\frac{dy}{dx} = xy$ can't be solved the same way: y on the right

need to know y in terms of x → solution

notice if we treat $\frac{dy}{dx}$ as a fraction: dy over dx

we can divide by y and multiply by dx

$\frac{1}{y}\,dy = x\,dx$ we can now integrate both sides

$$\int \frac{1}{y}\,dy = \int x\,dx$$

$$\ln|y| + C_1 = \frac{1}{2}x^2 + C_2$$

goal: y as a function of x

if possible (explicit form)

if not, implicit

$$\ln|y| = \frac{1}{2}x^2 + \boxed{C_2 - C_1}$$

→ $C_2 - C_1$ is still constant

call it C

$$\ln|y| = \frac{1}{2}x^2 + C$$

→ implicit solution

$$y = e^{\frac{1}{2}x^2 + C} = e^{\frac{1}{2}x^2} \cdot \boxed{e^C}$$

C is a const. e is const.

so e^C is const.

call it C again

$$\boxed{y = Ce^{\frac{1}{2}x^2}}$$

explicit form

this kind of differential eq. is called <u>separable</u>

→ separate x and y by multiplication or division can't separate this: $\dfrac{dy}{dx} = x^2 + y^2$

$$\frac{dy}{dx} - y^2 = x^2$$

no addition/sub

mult. by dx

$$dy - y^2\,dx = x^2\,dx$$

division doesn't work either

$$\frac{1}{x^2 + y^2}\,dy = dx$$

can't "move" it to the x side

NOT all differential eqs. are separable

<u>example</u>

$$(1+x)y' = y$$

$$(1+x)\frac{dy}{dx} = y$$

$$\frac{1}{y}\,dy = \frac{1}{1+x}\,dx \qquad \text{seperated.}$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{1+x}\,dx$$

$$\ln|y| = \ln|1+x| + C \qquad \text{~~explici~~ implicit}$$

$$e^{\ln|y|} = e^{\ln|1+x|+C} = e^{\ln|1+x|}\cdot e^{C} = e^{\ln|1+x|}\cdot C$$

$$y = (1+x)\cdot C$$

$$\boxed{y = C(1+x)} \qquad \text{explicit form}$$

example

$y' = xy^2 - y^2 \qquad y(1) = 2$

$$\frac{dy}{dx} = xy^2 - y^2 = y^2(x-1)$$

$$\frac{1}{y^2}\,dy = (x-1)\,dx$$

$$\int \frac{1}{y^2}\,dy = \int (x-1)\,dx$$

$$-\frac{1}{y} = \frac{1}{2}x^2 - x + C$$

we can solve for C now

$y(1) = 2 \rightarrow$ when $x = 1$ $y = 2$

$$-\frac{1}{2} = \frac{1}{2} - 1 + C$$

$$-\frac{1}{2} = -\frac{1}{2} + C \qquad \text{so} \quad C = 0$$

$$-\frac{1}{y} = \frac{1}{2}x^2 - x \qquad \boxed{y = -\frac{1}{\frac{1}{2}x^2 - x}}$$

or we can solve for "c" after finding explicit solution

$$-\frac{1}{y} = \frac{1}{2}x^2 - x + c$$

$-1 \cdot c$

multiply by -1: $\frac{1}{y} = -\frac{1}{2}x^2 + x + c$

$$y = \frac{1}{-\frac{1}{2}x^2 + x + c} \qquad y(1) = 2$$

$$2 = \frac{1}{+\frac{1}{2} + C} \qquad \frac{1}{2} + c = \frac{1}{2} \qquad c = 0$$

$$y = \frac{1}{-\frac{1}{2}x^2 + x} \qquad \text{same!}$$

separable shows up in a lot of applications

population: $\frac{dp}{dt} = kp$ $\qquad$ $p(t)$: population at t

growth constant $(k > 0)$

radioactive decay: $\frac{dp}{dt} = kp$ $\qquad$ $p(t)$: amount of something

decay constant $(k < 0)$

for uranium 235 $k = -3.12 \times 10^{-17}$

Newton's Law of Cooling : $\frac{dT}{dt} = K(M - T)$

temp

time

surrounding

Solve this: big T and little t are variables like x and y
rest are constants

$$\int \frac{1}{M-T}\,dT = \int K\,dt$$

$$-\ln|M-T| = Kt + C$$

$$\ln|M-T| = -Kt + C$$

$$M-T = e^{-Kt}\cdot e^{C} = Ce^{-Kt}$$

$$\boxed{T = M - Ce^{-Kt}} \qquad \lim_{t\to\infty} T = M$$

1.5 Linear First-Order Differential Eq.

eq. of this form: $\dfrac{dy}{dx} + \underline{\underline{P(x)}}\, y = \underline{\underline{Q(x)}}$

cannot contain y

$y' + xy = \cos x$ is linear: $P(x) = x$, $Q(x) = \cos x$

$y' - y^2 = 1$ is NOT: $y' \underbrace{- y}\, y = 1$

P contains y

how to solve linear 1st-order?

$$\frac{dy}{dx} + \frac{1}{x} y = 0$$

multiply $\underline{\underline{x}}$: $\underbrace{x \dfrac{dy}{dx} + y}$ $= 0$

integrating factor

$$\frac{d}{dx}(xy) = 0$$

So, $xy = C$ because its deriv. is 0

$$\boxed{y = \frac{C}{x}}$$

ALL linear 1st-order can be solved by multiplying by an integrating factor, turn left into derivative of a product

Then solve by integration

How to find integrating factor

$$y' + p(x)y = Q(x)$$

the integrating factor is $I = e^{\int p(x)dx}$

example $\quad y' - 4y = e^{4x}$

no y, so linear

integrating factor: $I = e^{\int P(x)dx} = e^{\int -4\,dx}$

$I = e^{-4x}$

ANY constant of integration works, for simplicity we normally choose 0

multiply both sides by I

$$e^{-4x}(y' - 4y) = e^{4x} \cdot e^{-4x}$$

$$\underbrace{e^{-4x}y' - 4e^{-4x}y}_{} = 1 \quad \rightarrow \quad e^{4x} \cdot e^{-4x} = e^{4x-4x} = e^0$$

$$\underbrace{\frac{d}{dx}\left(e^{-4x}y\right)}_{\text{ALWAYS } \frac{d}{dx}(Iy)} = 1 \quad \rightarrow \quad \frac{d(e^{-4x}y)}{dx} = 1$$

$$d(e^{-4x}y) = dx$$

$$e^{-4x} y = \int 1 \cdot dx$$

$$= x + c$$

THIS c is important

$$y = (x + c) e^{4x}$$

$$\boxed{y = x e^{4x} + c e^{4x}}$$

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why does the C in I not matter?

$$y' - 4y = e^{4x}$$

$p(x) = -4$ $\quad I = e^{\int -4\,dx} = e^{-4x+c} = e^{-4x} \cdot e^{c}$

$$I = C e^{-4x}$$

1 if "c" is 0

multiply by I on both sides

$$C e^{-4x}(y' - 4y) = e^{4x} \cdot C e^{-4x}$$

$$C\,(e^{-4x} y' - 4 e^{-4x} y) = C$$

can divide out

terminal velocity: $\frac{dv}{dt} = g - \frac{c}{m}v$

for ease of calculation, let $g = 10$ $\frac{c}{m} = 2$

$$\frac{dv}{dt} = 10 - 2v$$

separable? yes.

$$\frac{1}{10-2v}\,dv = dt$$

linear? yes.

$$\frac{dv}{dt} + 2v = 10$$

"standard form" of linear eq.

let's solve it as linear

$$I = e^{\int 2\,dt} = e^{2t}$$

$$e^{2t}\left(\frac{dv}{dt} + 2v\right) = 10 \cdot e^{2t}$$

$$\frac{d}{dt}\left(e^{2t}v\right) = 10e^{2t}$$

$$e^{2t} v = \int 10 e^{2t}\, dt = 5e^{2t} + C$$

$$\boxed{v(t) = 5 + Ce^{-2t}}$$

terminal velocity

C: depend on initial condition

$v(0) = v_0$