1.6 Substitution Methods and Exact Equations

we've covered: separable and first-order linear

NOT all equations are separable or linear, for example $y' = (x+y)^2$

but some can be turned into separable or linear w/ a substitution.

example $y' = (x+y)^2$

sub: let $v = x+y$

now rewrite in terms of v

$v' = 1 + y' \rightarrow y' = v' - 1$

$v' - 1 = v^2$

$\boxed{v' = v^2 + 1}$ separable in v and x

$$\frac{dv}{dx} = v^2 + 1$$

$$\frac{1}{v^2+1}\,dv = dx$$

integrate

$$\tan^{-1}(v) = x + C$$

$$v = \tan(x+C)$$

now undo the sub $v = x+y$

$$x+y = \tan(x+C)$$

$$\boxed{y = \tan(x+C) - x}$$

this technique works for all eqs of the form

$$\frac{dy}{dx} = f(ax+by+c)$$

$$v = ax+by+c$$

a first-order equation is called homogeneous if it can be written as $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

can mean different things in different contexts

for example, $y' = \frac{2x-y}{x+7y}$

divide top and bottom by x

$$y' = \frac{\frac{2x-y}{x}}{\frac{x+7y}{x}} = \frac{2-\left(\frac{y}{x}\right)}{1+7\left(\frac{y}{x}\right)}$$

function of $\frac{y}{x}$ so eq. is homogeneous

with an appropriate substitution, it becomes separable or linear

for homogeneous eqs, make sub: $v = \frac{y}{x}$ $(x \neq 0)$

eliminate y and write eq in terms of v and x

$$y' = \frac{2-\left(\frac{y}{x}\right)}{1+7\left(\frac{y}{x}\right)}$$

$v = \frac{y}{x} \rightarrow y = vx$ function of x

product rule $y' = v + xv'$

$$v + xv' = \frac{2-v}{1+7v}$$

Subtract v

$$xv' = \frac{2-v}{1+7v} - v = \frac{2-v}{1+7v} - \frac{v(1+7v)}{1+7v} = \frac{2-v-v-7v^2}{1+7v}$$

$$\boxed{xv' = \frac{2-2v-7v^2}{1+7v}}$$

separable in v and x

solve for v, undo sub $v = \frac{y}{x}$

$$\frac{1+7v}{2-2v-7v^2}\,dv = \frac{1}{x}\,dx$$

$$\int \frac{1+7v}{2-2v-7v^2}\,dv = \int \frac{1}{x}\,dx$$

$u = 2 - 2v - 7v^2$

$du = (-2 - 14v)\,dv$

$= -2(1 + 7v)\,dv$

$$-\frac{1}{2}\int \frac{1}{u}\,du = \int \frac{1}{x}\,dx$$

$-\frac{1}{2}\ln u = \ln x + c$

$\ln u = -2\ln x + c$

$u = e^{-2\ln x + c} = e^{\ln x^{-2}} \cdot e^{c} = C x^{-2}$

$2 - 2v - 7v^2 = C x^{-2}$

$2 - 2\left(\frac{y}{x}\right) - 7\left(\frac{y}{x}\right)^2 = C x^{-2}$

multiply both sides by x^2

$$\boxed{2x^2 - 2xy - 7y^2 = C}$$

Bernoulli differential equation

$$y' + P(x)\,y = Q(x)\,y^n \qquad n \neq 0,\ n \neq 1$$

could become separable and/or linear

eq. becomes linear

make this substitution: $v = y^{1-n}$

BIG picture

solve for v'

eliminate y, y'

solve for v

undo subs

example

$$x^2 y' + xy = y^2$$

put in standard form (coefficient of y' is 1)

$$y' + \frac{1}{x}y = \frac{1}{x^2}y^{2} \to n$$

$$v = y^{1-n} = y^{1-2} = y^{-1}$$

$$v' = -y^{-2}y' \qquad \text{so} \quad y' = -y^2 v'$$

$$y = \frac{1}{v}$$

$$y' + \frac{1}{x} y = \frac{1}{x^2} y^2 \longrightarrow -y^2 v' + \frac{1}{x} y = \frac{1}{x^2} y^2$$

$$\cancel{-y^2 v' + \frac{1}{x}} \frac{1}{v} = \qquad \text{divide by } -y^2$$

$$v' - \frac{1}{x} \left(\frac{1}{y}\right) = -\frac{1}{x^2}$$

$$v = y^{-1} = \frac{1}{y}$$

$$v' - \frac{1}{x} v = -\frac{1}{x^2} \qquad \text{linear in } v$$

$$I = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = x^{-1}$$

$$\frac{d}{dx}\left(x^{-1} v\right) = -\frac{1}{x^3}$$

$$x^{-1} v = \frac{1}{2x^2} + c$$

$$v = \frac{1}{2x} + cx = y^{-1} \qquad v = y^{-1}$$

$$\boxed{y = \frac{2x}{1 + cx^2}}$$

Exact Differential Eq

of the form $M\,dx + N\,dy = 0$

where $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

solution is $f(x,y) = C$ where $M = \dfrac{\partial f}{\partial x}$ and $N = \dfrac{\partial f}{\partial y}$

Example $\underbrace{(3x^2+2y^2)}_{M}\,dx + \underbrace{(4xy+6y^2)}_{N}\,dy = 0$

eq. is exact if $M_y = N_x$

check: $M_y = \dfrac{\partial}{\partial y}(3x^2+2y^2) = 4y$

$N_x = \dfrac{\partial}{\partial x}(4xy+6y^2) = 4y$

equation is exact, so solution is $\underbrace{f = C \text{ where } M = f_x,\ N = f_y}_{\text{true only if exact}}$

$$M = 3x^2 + 2y^2 = \frac{\partial f}{\partial x} \quad - (1)$$

$$N = 4xy + 6y^2 = \frac{\partial f}{\partial y} \quad - (2)$$

integrate (1) with respect to x

$$f = \int (3x^2 + \underset{\text{constant}}{2y^2})\,dx = x^3 + 2y^2x + \overbrace{g(y)}^{\text{"constant" as far as } \frac{\partial}{\partial x} \text{ is concerned}} \quad - (3)$$

take partial of (3) with respect to $y \rightarrow \frac{\partial f}{\partial y}$

it should be equal to (2)

$$\underbrace{\frac{\partial f}{\partial y} = 4xy + \frac{dg}{dy}}_{\text{from } \frac{\partial}{\partial y} \text{ of } (3)} = \underbrace{4xy + 6y^2}_{\text{from } N\ (2)}$$

$$\frac{dg}{dy} = 6y^2 \rightarrow g = 2y^3 + C_1$$

solution is $f(x,y) = C$

$f = x^3 + 2y^2x + 2y^3 + C_1$

so solution is $\boxed{x^3 + 2y^2x + 2y^3 = C}$

some 2nd-order eqs. can be transformed into a 1st-order that we can solve (separable, linear, homogeneous, or exact)

these are the two types: no x on either side

no y on either side (but can have y' and y'')

<u>example</u> $y'' = -y$ no x on either side

make this sub: $p = \frac{dy}{dx} = y' \longrightarrow p$

$$y'' = \frac{dp}{dx} = \frac{dp}{dy}\boxed{\frac{dy}{dx}} = p\frac{dp}{dy}$$

Big picture: remove x from the eq.

$$y'' = -y$$

$$p\frac{dp}{dy} = -y$$

separable in p and y

solve for p, then solve for y from $p = \frac{dy}{dx}$

example

$$xy'' + y' = x$$

make this sub: $p = \frac{dy}{dx} = y'$

$$\frac{dp}{dx} = y''$$

$$x\frac{dp}{dx} + p = x$$

$$\frac{dp}{dx} + \frac{1}{x}p = 1$$

1st-order linear in p and x

big picture: solve for p, then y from $p = \frac{dy}{dx}$