3.1 + 3.2 Determinants

"HW 13" + "HW 14" due together

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\det(A) = ad - bc$

what about 3×3 and beyond?

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \sim \cdots \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}$$

where

$$\Delta = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

if $\Delta \neq 0$, then A^{-1} exists, just like 2×2 case when $\det(A) \neq 0$

Δ <u>is</u> the determinant of 3×3 A.

rewrite Δ

$$\Delta = a_{11}\underbrace{(a_{22}a_{33} - a_{23}a_{32})}_{\text{determinant of } \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}} - a_{12}\underbrace{(a_{21}a_{33} - a_{23}a_{31})}_{\text{det of } \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}} + a_{13}\underbrace{(a_{21}a_{32} - a_{22}a_{31})}_{\text{det of } \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}}$$

A w/ first row and first col ~~both~~ covered

A w/ 2nd col and 1st row covered

A w/ 3rd col and 1st row covered

example

$$A = \begin{bmatrix} 2 & 0 & 4 \\ 3 & 4 & 2 \\ 0 & 4 & -2 \end{bmatrix} \qquad \det(A) = ? \qquad (\det A)$$

$$\det(A) = \begin{vmatrix} 2 & 0 & 4 \\ 3 & 4 & 2 \\ 0 & 4 & -2 \end{vmatrix}$$

sign change at every even col or even row

$$= (2)\begin{vmatrix} 4 & 2 \\ 4 & -2 \end{vmatrix} - (0)\begin{vmatrix} 3 & 2 \\ 0 & -2 \end{vmatrix} + (4)\begin{vmatrix} 3 & 4 \\ 0 & 4 \end{vmatrix}$$

"cofactor expansion"

$$= (2)(-8-8) - (0)(-6-0) + (4)(12-0) = 16$$

we can do cofactor expansion along any row or column.

last example : along first row

try along 2nd column

$$A = \begin{bmatrix} 2^{+} & 0^{-} & 4^{+} \\ 3^{-} & 4^{+} & 2^{-} \\ 0^{+} & 4^{-} & -2^{+} \end{bmatrix}$$

$$\det(A) = -(0)\begin{vmatrix} 3 & 2 \\ 0 & -2 \end{vmatrix} + (4)\begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} - (4)\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}$$

$$= (4)(-4) - (4)(-8) = 16$$

another way for 3x3 : Rule of Sarrus

$$\begin{array}{ccc|cc} & & - & - & - \\ \hline 2 & 0 & 4 & 2 & 0 \\ 3 & 4 & 2 & 3 & 4 \\ 0 & 4 & -2 & 0 & 4 \\ \hline & & + & + & + \end{array}$$

copies of 1st and 2nd cols

$$\det(A) = (2)(4)(-2) + (0)(2)(0) + (4)(3)(4)$$

$$- (0)(4)(4) - (4)(2)(2) - (-2)(3)(0) = 16$$

4x4 and beyond are just series of 3x3's

<u>example</u>

$$\begin{vmatrix} 5^{+} & 0^{-} & 0^{+} & 4^{-} \\ 2^{+} & 7^{+} & 3^{-} & -8^{+} \\ 2^{+} & 0^{-} & 0^{+} & 0^{-} \\ 8^{-} & 3^{+} & 1^{-} & 9^{+} \end{vmatrix}$$

cofactor expansion along ANY column or row, but row / col with lots of zeros are best

$$= (2)\begin{vmatrix} 0^{+} & 0^{-} & 4^{+} \\ 7 & 3 & -8 \\ 3 & 1 & 9 \end{vmatrix} + (0)\begin{vmatrix} \text{I don't care} \end{vmatrix} + (0)\begin{vmatrix} \text{IDC} \end{vmatrix} + (0)\begin{vmatrix} \text{IDC} \end{vmatrix}$$

$$= (2)\left\{ (0)\begin{vmatrix} \text{IDC} \end{vmatrix} - (0)\begin{vmatrix} \text{IDC} \end{vmatrix} + (4)\begin{vmatrix} 7 & 3 \\ 3 & 1 \end{vmatrix} \right\}$$

$$= (2)(4)(7-9) = -16$$

If A is triangular then $\det(A)$ is product of main diagonal elements

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = 3$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} = (1)\begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = (1)(4)(6) = 24$$

How row operations affect determinants

(1) each time two rows are interchanged, the determinant changes sign

(2) if one row of A is multiplied by K to produce B, then $\det(B) = K \det(A)$

(3) if multiples of one row is added to another, the determinant does not change

<u>example</u> $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ $\det(A) = 3$

interchange rows: $\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$ $\det = -3$

<u>example</u> $A = \begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ 3 & 4 \end{bmatrix}$ $\det(A) = -\frac{1}{5}$

factor out $\frac{1}{10}$ out of first row: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$$

$$\left(\frac{1}{10}\right)(-2) = -\frac{1}{5}$$

Just as in 2×2, any square matrix A is invertible if and only if $\det(A) \neq 0$

Properties of determinants

$\det(A^T) = \det(A)$ 　　this means we can do elementary <u>column</u> ops

$\det(AB) = \det(A)\det(B)$

but $\det(A+B) \neq \det(A) + \det(B)$ in general

$\det(A^n) = [\det(A)]^n$

linearity property : if one col of A is multiplied by some c, then determinant is also multiplied by c

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ 　　$\det(A) = -2$

$B = \begin{bmatrix} 2 & 2 \\ 6 & 4 \end{bmatrix}$ 　　$\det(B) = -4$

col 1 of A times 2

if one col A is linear combo of column vectors, then determinant is the linear combo of determinants of matrices w/ those vectors

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \det(A) = -2$$

$$B = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \qquad \det(B) = 0$$

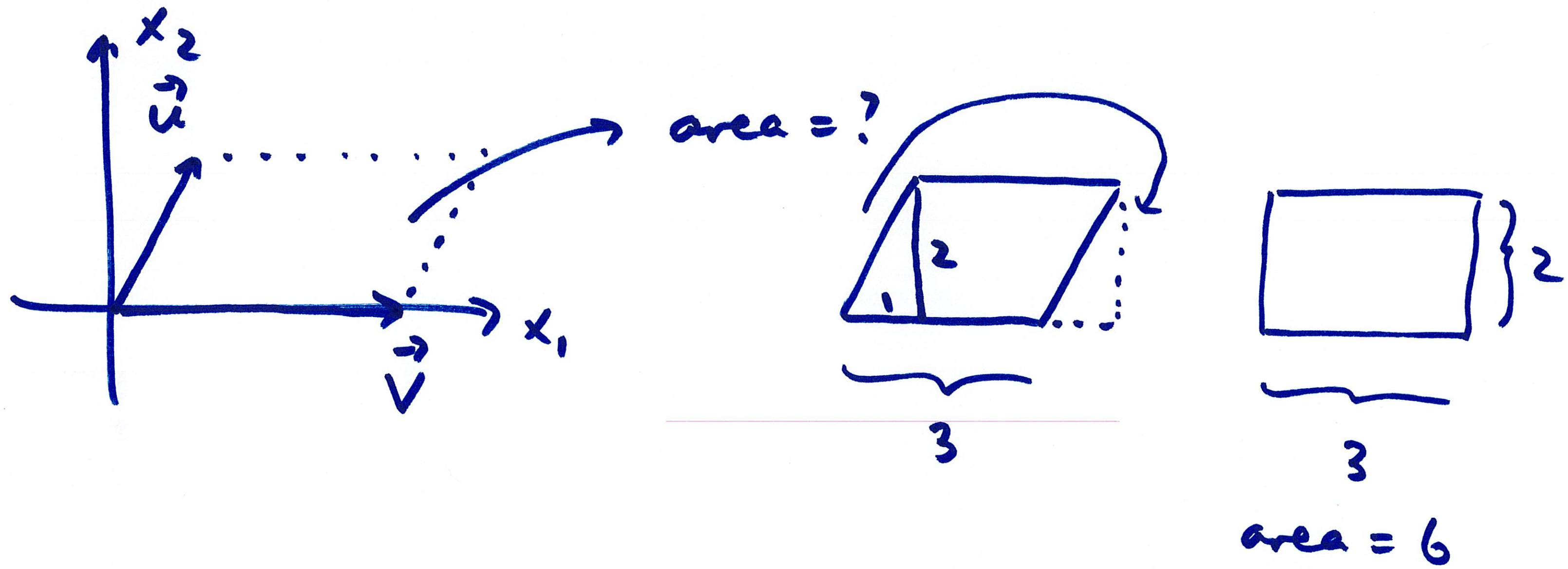
$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) + \det\left(\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}\right)$$

$$= -2 + 2 = 0$$

what is the determinant?

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$



$$A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$$

$$|\det(A)| = |0 - 6| = 6$$