Exam 1

Average : 74

A: 86

B: 73

C: 60

D: 50

4.5 The Dimension of a Vector Space

4.4 : a vector space V with basis B containing n vectors is isomorphic to $\mathbb{R}^n$

example : the standard basis of $\mathbb{P}_3$ is $B = \{1, t, t^2, t^3\}$

a typical element in $\mathbb{P}_3$ is $\vec{p}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$

it behaves and acts like a vector in $\mathbb{R}^4$ $\vec{p} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$$\vec{p}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$\vec{q}(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3$$

$$\vec{p} + \vec{q} = (a_0 + b_0) + (a_1 + b_1) t + (a_2 + b_2) t^2 + (a_3 + b_3) t^3$$

same as $\vec{p} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$ $\vec{q} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$$\vec{p} + \vec{q} = \begin{bmatrix} a_0 + b_0 \\ a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}$$

same for scaler multiplication

If a vector space V has a basis $\mathcal{B} = \{\vec{b_1}, \dots, \vec{b_n}\}$, then any set in V containing more than n vectors must be linearly dependent

why? each $\vec{b_i}$ is $n \times 1$

if we have m vectors, where $m > n$, then the matrix $A = [\vec{b_1}, \dots, \vec{b_m}]$ has n rows and m columns, and A ~~can have~~ has n pivots

but there are m columns, so there are free variables in $A\vec{x} = \vec{0}$ $\longrightarrow$ dependent set.

If a vector space V has a basis B of n vectors, then every basis of V must also have n vectors.

why? If $B = \{\vec{b}_1, \dots, \vec{b}_n\}$ is a basis, then we know the $\vec{b}_i$ are linearly independent and span V.

If $C = \{\vec{c}_1, \dots, \vec{c}_m\}$ is another basis of V. We know $\vec{c}_i$ must be linearly independent, so knowing there are <u>at most</u> n linearly independent vectors in V (from B), we know $m \leq n$.

The Basis Theorem

But since B, a basis, spans V, we know we need no fewer than n vectors to span V. So now know C, being a basis, must have no fewer than n vectors. So $m \geq n$. So, $m = n$.

{this means ANY other set of n linearly vectors in V is a basis.
" " " " " " vectors spanning V is a basis.

The number of basis vectors is called the <u>dimension</u> of the vector space. Vector spaces can be finite-dimensional or infinite-dimensional.

example: all continuous functions.

We already know about $\underbrace{\dim \operatorname{Col} A}_{\#\text{ of basic variables}}$ and $\underbrace{\dim \operatorname{Nul} A}_{\#\text{ of free variables}}$

example: How many dimensions does a subspace of all $\mathbb{P}_{10}$ whose 5^{th} and 7^{th} coefficients are the same have?

10. Because $\mathbb{P}_{10}$ has 11 coefficients, but we can only freely choose 10 of them.

<u>example</u> The first 3 Chebyshev polynomials are $1,\ t,\ 2t^2-1$. Can these form a basis for $\mathbb{P}_2$?

$\mathbb{P}_2$ is 3-dimensional, ~~and so~~ $a_0 + a_1 t + a_2 t^2$.

The standard basis: $\{1, t, t^2\}$

$\{1, t, 2t^2-1\}$ has 3 vectors, and we know from earlier results if these vectors are linearly ~~indpe~~ independent, then they must for a basis.

rewrite as vectors in $\mathbb{R}^3$ using $\{1, t, t^2\}$ as "coordinates"

$$\{1,\ t,\ t^2\} \to \begin{bmatrix}1\\0\\0\end{bmatrix},\ \begin{bmatrix}0\\1\\0\end{bmatrix},\ \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\{1,\ t,\ 2t^2-1\} \to \begin{bmatrix}1\\0\\0\end{bmatrix},\ \begin{bmatrix}0\\1\\0\end{bmatrix},\ \begin{bmatrix}-1\\0\\2\end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{2} \end{bmatrix}$$

3 pivots, 3 vectors, so linearly independent

so $\{1, t, 2t^2-1\}$ must be a basis for $\mathbb{P}_2$

Spanning Set Theorem allows us to make basis by throwing out linearly dependent ones: e.g. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

is NOT basis for $\mathbb{R}^2$

but $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

is.

~~we~~ Likewise, we can add linearly independent vectors to make basis for a subspace

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

add $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (indp from existing ones)

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is basis for $\mathbb{R}^2$