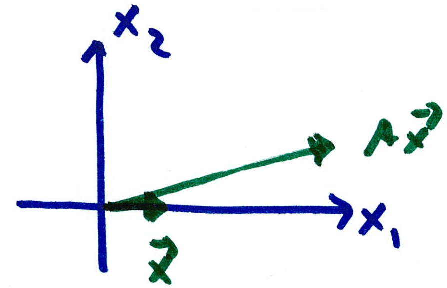
5.1 + 5.2 Eigenvectors, eigenvalues, and Characteristic equations

HW 21 + HW 22 due together

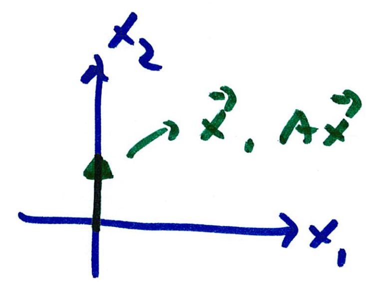
$A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$ $\quad A\vec{x}$ is a transformation of $\vec{x}$



$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

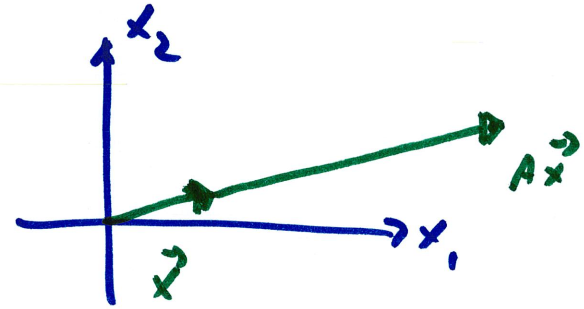
more than likely both magnitude and direction will change

however, some do not change direction



$$\begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



these are called <u>eigenvectors</u>

the scaling factors after transformation are called <u>eigenvalues</u>

e.g. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$

with the corresponding ~~value~~ eigenvalue of 5

$$\boxed{\text{this means if } \vec{x} \text{ is an eigenvector of } A \text{ w/ corresponding eigenvalue } \lambda, \text{ then } A\vec{x} = \lambda\vec{x}}$$

<u>example</u> Is $\vec{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ an eigenvector of $A = \begin{bmatrix} -1 & 4 \\ 3 & 3 \end{bmatrix}$?

$$A\vec{x} = \begin{bmatrix} -1 & 4 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix} = \underset{\lambda = -3}{-3} \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\vec{x}}$$

How to find eigenvector if we know eigenvalue?

<u>example</u>

$$A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \qquad \text{eigenvalues : } 1,\ 5$$

eigenvector for $\lambda = 1$:

$$A\vec{x} = \lambda \vec{x}$$

$$A\vec{x} - \lambda \vec{x} = \vec{0}$$

$$(A - \underset{\uparrow\, 1}{\lambda} I)\underset{\downarrow\, ?}{\vec{x}} = \vec{0} \qquad \Rightarrow \text{homogeneous eq.}$$

$$\left[\begin{array}{cc|c} 7-1 & 4 & 0 \\ -3 & -1-1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 6 & 4 & 0 \\ -3 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 6 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

eigenvector $\neq \vec{0}$
must have nontrivial solution

x_2 free

$x_1 = -\frac{2}{3} x_2$

$$\vec{x} = \begin{bmatrix} -2/3 \\ 1 \end{bmatrix} x_2$$

make this ANY convenient # $\neq 0$

let $x_2 = 3$.

$$\boxed{\vec{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix},\ \lambda = 1}$$

$\lambda = 5$

$(A - \lambda I)\vec{x} = \vec{0}$ $\begin{bmatrix} 2 & 4 & 0 \\ -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

x_2 free $x_1 = -2x_2$

$\underbrace{\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$ $\boxed{\vec{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \lambda = 5}$

is a vector space with basis $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

called <u>eigenspace</u>

↳ all multiples of eigenvector and the zero vector

How to find eigenvalues?

Special case: if A is triangular, then the main diagonal elements are eigenvalues.

why? $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ $\quad A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix}$

if λ is an eigenvalue, then $(A - \lambda I)\vec{x} = \vec{0}$ has nontrivial solutions (eigenvectors)

so λ must be equal to 1, 4, or 6 in above A (to <u>not</u> have rank of 3)

eigenvector CANNOT be zero vector

eigenvalue CAN be zero.

if $\lambda = 0$, then $\cancel{(A-\lambda I)x}$ $A\vec{x} = \lambda\vec{x}$ becomes

$A\vec{x} = \vec{0}$ and $(A - \lambda I)\vec{x} = 0$ has <u>nontrivial</u> solutions

$\Rightarrow$ A^{-1} does NOT exist (because $\det A = 0$, columns of A are not independent, etc) also, all eigenvectors corresponding to distinct eigenvalues <u>are</u> linearly independent.

How to find eigenvalues in general case

$$A\vec{x} = \lambda \vec{x}$$

$(A - \lambda I)\vec{x} = \vec{0}$ <u>must</u> have nontrivial solutions

$$\Rightarrow \boxed{\det(A - \lambda I) = 0}$$

↳ solve for λ

<u>example</u> $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ $\lambda = ?$

$$A - \lambda I = \begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = \overbrace{(5-\lambda)^2 - 9}^{\substack{\text{characteristic eq.} \\ (n \times n \text{ matrix} \to n^{th}\text{-degree polynomial})}} = 0$$

$$(5-\lambda) = 3 \quad \text{or} \quad (5-\lambda) = -3$$

$$\lambda = 2 \quad \text{or} \quad \lambda = 8$$

2×2 $A \Rightarrow 2$, possibly repeated or complex λ

then find eigenvectors by following earlier example.

<u>example</u>

$$A = \begin{bmatrix} -8 & -1 \\ 1 & -6 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -8-\lambda & -1 \\ 1 & -6-\lambda \end{vmatrix} = 0$$

$$(-8-\lambda)(-6-\lambda) + 1 = 0$$

$$\lambda^2 + 14\lambda + 49 = 0$$

algebraic multiplicity

$\lambda = 7, 7$ $\quad$ 7 is repeated, shows up twice example

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -3 & 2 & -2 \\ 0 & 7 & 0 \end{bmatrix}$$

row reduction changes eigenvalues

do NOT do ERO's before finding λ's

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ -3 & 2-\lambda & -2 \\ 0 & 7 & -\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} 2-\lambda & -2 \\ 7 & -\lambda \end{vmatrix} + (1)\begin{vmatrix} -3 & 2-\lambda \\ 0 & 7 \end{vmatrix} = 0$$

$$(1-\lambda)\left[(2-\lambda)(-\lambda)+14\right] + (-3)(7) = 0$$

$$\ldots \quad -\lambda^3 + 3\lambda^2 - 16\lambda - 7 = 0$$