5.3 (1) Diagonalization

exam 2 covers up to this lesson

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, in general $A^k \neq \begin{bmatrix} a^k & b^k \\ c^k & d^k \end{bmatrix}$

but if $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, then $A^k = \begin{bmatrix} a^k & 0 \\ 0 & d^k \end{bmatrix}$

why? $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ $A^3 = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$

$$= \begin{bmatrix} a^2 & 0 \\ 0 & d^2 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} a^3 & 0 \\ 0 & d^3 \end{bmatrix}$$

If A is a square matrix, then it is <u>diagonalizable</u> if it is <u>similar</u> to a diagonal matrix D. This means there exists matrix P such that

$$A = PDP^{-1}$$

what are D and P?

$A = PDP^{-1}$

$AP = PD\underbrace{P^{-1}P}_{I} \Rightarrow AP = PD$

let $P = [\vec{v}_1 \ \ \vec{v}_2]$ $\quad D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$

$AP = PD$ is then $A[\vec{v}_1 \ \ \vec{v}_2] = [\vec{v}_1 \ \ \vec{v}_2]\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$

$$[A\vec{v}_1 \ \ A\vec{v}_2] = [d_1\vec{v}_1 \ \ d_2\vec{v}_2]$$

this means $A\vec{v}_1 = d_1\vec{v}_1$ and $A\vec{v}_2 = d_2\vec{v}_2$

so, $\vec{v}_1$ is an eigenvector of A w/ the corresponding eigenvalue d_1

$\vec{v}_2$ is an eigenvector of A w/ the corresponding eigenvalue d_2

($3\times3 \rightarrow 3$ eigenvalue/vector pairs
$n\times n \rightarrow n$ " " ")

P = matrix w/ eigenvectors as columns

D = " " corresponding eigenvalues on the main diagonal

if eigenvalues are distinct, then the matrix is always diagonalizable, but might be so if eigenvalues are repeated.

<u>example</u>

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

triangular, so eigenvalues are on the main diagonal

$\lambda = 1, \quad \lambda = 3$

find eigenvector for $\lambda = 1$:

$$(A - \lambda I)\vec{x} = \vec{0} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2 free $\quad x_1 = -x_2$

$\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ eigenvector: $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = 1$

repeat for $\lambda = 3$

$(A-\lambda I)\vec{x} = \vec{0}$ $\begin{bmatrix} -2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

x_2 free $x_1 = 0$

$\vec{x} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ eigenvector : $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\lambda = 3$

$P = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} -1 & 0 \\ +1 & +1 \end{bmatrix}$

(or $P = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$)

$A = PDP^{-1}$

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

if $A = PDP^{-1}$

then $A^k = (PDP^{-1})^k$

$$= (PD\underbrace{\boxed{P^{-1})\,(P}}_{I}D\underbrace{\boxed{P^{-1})\,(P}}_{I}D\underbrace{\boxed{P^{-1})\,\cdots}}_{I}\underbrace{\boxed{(P}}_{I}DP^{-1}) \quad k \text{ times}$$

$$A^k = PD^kP^{-1}$$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ +1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^4 & 0 \\ 0 & 3^4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 81 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 1 & 81 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 80 & 81 \end{bmatrix}$$

What if eigenvalues are repeated?

Ideal case: dimension of eigenspace = algebraic multiplicity

(# of eigenvectors found the normal way

= # of times the eigenvalue appears)

Bad case: dimension of eigenspace < algebraic multiplicity

$\Rightarrow$ matrix is NOT diagonalizable

<u>example</u>

$$A = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 5 & 2 \\ 2 & 2 & 5 \end{bmatrix} \qquad \lambda = 3, 3, 9$$

find eigenvectors for $\lambda = 3$: (algebraic multiplicity = 2)

$$(A - \lambda I)\vec{x} = \vec{0}$$

$$\begin{bmatrix} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_2, x_3 free $\qquad x_1 = -x_2 - x_3$

geometric multiplicity = 2

$$\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

eigenvectors: $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

dim eigenspace = 2

repeat for $\lambda = 9$

$$\begin{bmatrix} -4 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 5 & 2 \\ 2 & 2 & 5 \end{bmatrix} = \underset{P}{\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix}} \underset{P^{-1}}{\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}}$$