1.2 Row Reduction and Echelon Forms

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$$

$-3R_1 + R_2$

$-5R_1 + R_3$

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix}$$

$-2R_2 + R_3$

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & -10 \end{bmatrix}$$

this is now in

Row Echelon Form

↓

ladders / stairs

this matrix is now in row echelon form because

1. any zero row is at bottom

2. the leading entry of each row is in a column to the right of the leading entry of the row above it.

3. all entries ~~to~~ in a column below a leading entry are zeros.

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & -10 \end{bmatrix}$$

let's go further and make more entries zeros

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$-3R_2 + R_1$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$-3R_3 + R_2$

$2R_3 + R_1$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

this is now in

Reduced Row Echelon Form

this matrix is in reduced row echelon form because (must satisfy conditions 1-3 above)

4. leading entry of each nonzero row is 1
5. each leading entry is the only nonzero element in its column.

the reduced row echelon form of a matrix is unique

$$\begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

neither because zero row not at bottom

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

row echelon

$$\begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ reduced row echelon

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back to $\begin{bmatrix} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$ in reduced row echelon form

the locations of the leading 1's are called <u>pivot positions</u>

the columns containing leading 1's are <u>pivot columns</u>

original matrix: $\begin{bmatrix} \boxed{1} & 3 & 5 & 7 \\ 3 & \boxed{5} & 7 & 9 \\ 5 & 7 & 9 & \boxed{1} \end{bmatrix}$

the elements correspond to the pivot positions in reduced echelon form are the ones used to make other rows in their column zeros the row reduction algorithm can be done to ANY matrix, not just augmented matrix of system

in a system, the number of pivots (not in last column) corresponds to the number of basic variables in a system.

<u>example</u>

$$3x_1 - 4x_2 + 2x_3 = 0$$
$$-9x_1 + 12x_2 - 6x_3 = 0$$
$$-6x_1 + 8x_2 - 4x_3 = 0$$

row equivalent to

$$\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & -4/3 & 2/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

sys has 3 variables, 1 pivot (corresponds to x_1)

$\Rightarrow$ x_2 and x_3 are <u>free variables</u>

x_1 (pivot column) is a <u>basic variable</u>

solution: $x_1 - \frac{4}{3}x_2 + \frac{2}{3}x_3 = 0$

x_2 free (arbitrary)

x_3 free (")

parametric description: $x_1 = \frac{4}{3}x_2 - \frac{2}{3}x_3$

x_2 free

x_3 free

# of zero rows = # of free variables

example The augmented matrix of some system reduces to

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \begin{bmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix} \end{array}$$

3 variables, 3 pivots. One solution $(1, -1, 1)$

example The augmented matrix of some system reduces to

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

"overdetermined system"
more eqs. than variables

3 variables, 3 pivots BUT one in right most column

Row 3 $\rightarrow$ $0 = 1$ no solution
(inconsistent system)

eventhough there is a zero row, system is inconsistent, no solution any time there is a pivot in the right most column of the reduced echolon form of an augmented matrix of a system $\Rightarrow$ inconsistent system, no solution.