5.4 Eigenvectors and Linear Transformations

$T: V \to W$ is the same as $T(\vec{x}) = A\vec{x}$ for an $m \times n$ A matrix.

$$\begin{array}{ccc} V & \to & W \\ \downarrow & & \downarrow \\ \mathbb{R}^n & & \mathbb{R}^m \end{array}$$

If a basis for V is $\{\vec{b_1}, \vec{b_2}, \dots, \vec{b_n}\}$ and a basis for W is $\{\vec{c_1}, \vec{c_2}, \dots, \vec{c_m}\}$, then a vector $\vec{x}$ in V is $\vec{x} = r_1\vec{b_1} + r_2\vec{b_2} + \dots + r_n\vec{b_n}$

and $T(\vec{x}) = T(r_1\vec{b_1} + \dots + r_n\vec{b_n})$

$$= r_1 T(\vec{b_1}) + r_2 T(\vec{b_2}) + \dots + r_n T(\vec{b_n})$$

$$= \Big[[T(\vec{b_1})]_C \; [T(\vec{b_2})]_C \cdots [T(\vec{b_n})]_C \Big] \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} = M[\vec{x}]_B$$

B-coordinate of $\vec{x}$

transformation of $\vec{b_1}$ written using C-coordinates

matrix for T relative to B and C

<u>example</u>:

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \qquad T: \underset{\mathbb{R}^4}{V} \to \underset{\mathbb{R}^3}{W}$$

a basis for V is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \{\vec{b}_1, \dots, \vec{b}_4\}$

a basis for W is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$

column 1: $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \cdot \vec{c}_1 + 0 \cdot \vec{c}_2 + 0 \cdot \vec{c}_3 = [T(\vec{b}_1)]_C$

column 2: $T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix} = -4 \cdot \vec{c}_1 + 2 \cdot \vec{c}_2 + 0 \cdot \vec{c}_3 = [T(\vec{b}_2)]_C$

and so on.

<u>example</u>: Find the matrix for T relative to B and D

if $T: \mathbb{R}^3 \to \mathbb{R}^2$ and $T(\vec{b_1}) = 7\vec{d_1} - 8\vec{d_2}$

$T(\vec{b_2}) = -7\vec{d_1} - 4\vec{d_2}$

$T(\vec{b_3}) = -\vec{d_1}$

$$A = \begin{bmatrix} 7 & -7 & -1 \\ -8 & -4 & 0 \end{bmatrix}$$

$\uparrow$

column 1: $T(\vec{b_1})$ in new coordinates ($\vec{d}$'s)

$\hookrightarrow = 7\vec{d_1} - 8\vec{d_2} \rightarrow [T(\vec{b_1})]_C = \begin{bmatrix} 7 \\ -8 \end{bmatrix}$

example: If $T: V \to \mathbb{R}^2$ and

$$T(x_1 \vec{b_1} + x_2 \vec{b_2} + x_3 \vec{b_3}) = \begin{bmatrix} -2x_1 + 8x_2 + 2x_3 \\ -4x_1 - 5x_2 \end{bmatrix}$$

find the transformation matrix.

$\rightarrow$ need $T(\vec{b_1})$, $T(\vec{b_2})$, $T(\vec{b_3})$ as columns of the matrix

because T is linear

$$\rightarrow x_1 T(\vec{b_1}) + x_2 T(\vec{b_2}) + x_3 T(\vec{b_3}) = x_1 \begin{bmatrix} -2 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} 8 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

so, $A = \begin{bmatrix} -2 & 8 & 2 \\ -4 & -5 & 0 \end{bmatrix}$

example $T: \mathbb{P}_3 \to \mathbb{P}_3$ defined by

$$T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = a_1 + 2a_2 t + 3a_3 t^2$$

find the transformation matrix.

basis: $\{1, t, t^2, t^3\}$

$$T(a_0) = a_0 T(1) = a_0 \cdot 0 \longrightarrow T(1) = 0$$

$$T(a_1 t) = a_1 T(t) = a_1 \cdot 1 \longrightarrow T(t) = 1$$

$$T(a_2 t^2) = a_2 T(t^2) = a_2 \cdot 2t \longrightarrow T(t^2) = 2t$$

$$T(a_3 t^3) = a_3 T(t^3) = a_3 \cdot 3t^2 \longrightarrow T(t^3) = 3t^2$$

basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

so $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ → differentiation of $\mathbb{P}_3$

check: $T(1+2t+3t^2+4t^3) = 2+6t+12t^2$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 12 \\ 0 \end{bmatrix}$$

If A is diagonalizable, then $A = PDP^{-1}$

P is matrix w/ eigenvectors as columns

D is diagonal matrix w/ corresponding eigenvalues

so, $A\vec{x} = \underbrace{PDP^{-1}}_{}\vec{x}$

how to interpret this as transformations?

what does $P^{-1}\vec{x}$ mean?

$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ $\lambda = 1, 2$

$\lambda = 1,\ \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda = 2,\ \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$P[\vec{x}_B] = \vec{x}$

$\vec{x}_B$: using eigenvectors as basis

coordinates using eigenvectors as basis

e.g. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow (1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$[\vec{x}_B] = P^{-1}\vec{x}$ $\longrightarrow$ coordinates of $\vec{x}$ in the eigenvector space

$$\underbrace{P D P^{-1}}\vec{x}$$

transform to the eigenvector coordinate from

$D\,(P^{-1}\vec{x}) \rightarrow$ lengthens/shortens whatever follows it

↓

diagonal

finally, $P\,(DP^{-1}\vec{x})$ goes back to original basis
(away from eigenvector basis)

so, if A is diagonalizable, $A\vec{x}$ first goes to eigenvector frame, then lengthen/shorten according to eigenvalues, then leaves the eigenvector frame.

---

trace $(A) \rightarrow$ if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ then trace $(A) = 1+5+9 = 15$

trace (ABC) = trace (BCA) = trace (CAB)