Exam 2

Avg: 77

A	88
B	76
C	60
D	50

6.1 Inner Product, Length, and Orthogonality

If $\vec{u}, \vec{v}$ are in $\mathbb{R}^n$ $\quad \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

the inner product (or dot product) of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = [u_1 \; u_2 \cdots u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \qquad \text{a scalar}$$

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} \quad \vec{u} \cdot \vec{v} = [1 \; 2 \; 3] \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = 6 + 10 + 12 = 28$$

note $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

properties: $\quad \vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \quad c = 7$

$$\boxed{\begin{aligned} &(\vec{u} + \vec{v}) \cdot \vec{w} \\ &= \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} \end{aligned}}$$

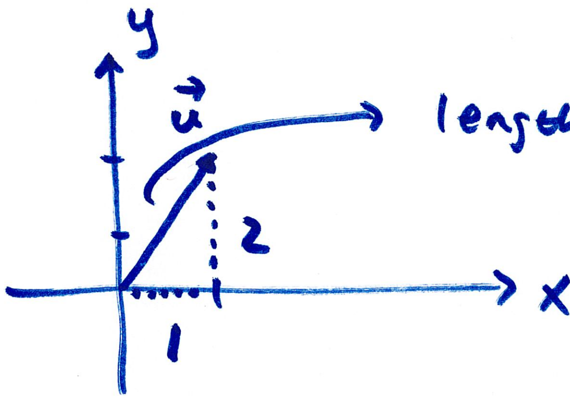
$$(\vec{u} + \vec{v}) \cdot \vec{w} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}^T \begin{bmatrix} 5 \\ 6 \end{bmatrix} = [4 \; 6] \begin{bmatrix} 5 \\ 6 \end{bmatrix} = 56$$

$$= \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} = [1 \; 2] \begin{bmatrix} 5 \\ 6 \end{bmatrix} + [3 \; 4] \begin{bmatrix} 5 \\ 6 \end{bmatrix} = 17 + 39 = 56$$

$$(c\vec{u})\cdot\vec{v} = \begin{bmatrix} 7 & 14 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 21 + \underset{56}{\cancel{E4}} = \cancel{85}\ 77$$

$$= c(\vec{u}\cdot\vec{v}) = 7\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 7(3+8) = 77$$

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \vec{u}\cdot\vec{u} = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1+4 = 5$$



length $= \sqrt{1^2+2^2} = \sqrt{5}$

so $\boxed{\vec{u}\cdot\vec{u} = \underbrace{\|\vec{u}\|}_{\text{length or norm of } \vec{u}}{}^2}$ for $\vec{u}$ in $\mathbb{R}^n$

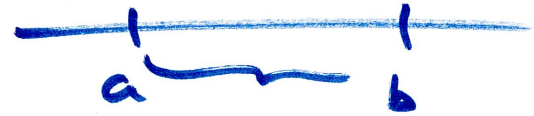
a vector is called a unit vector if its length is 1.

so a unit vector in the same direction as $\vec{u}$

is $\dfrac{\vec{u}}{\|\vec{u}\|} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

in $\mathbb{R}^1$, the distance between 2 numbers is $\cancel{\|a-b\| = \|}$

$$|b-a| = |a-b|$$



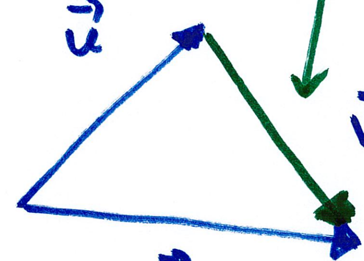
$$|b-a| = |a-b|$$

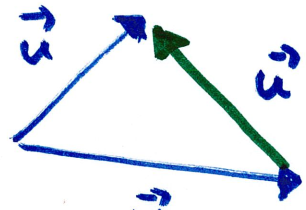
extend to $\mathbb{R}^n$

example: $\mathbb{R}^3$

distance between 2 vectors is

$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{v}-\vec{u}\| = \|\vec{u}-\vec{v}\|$$



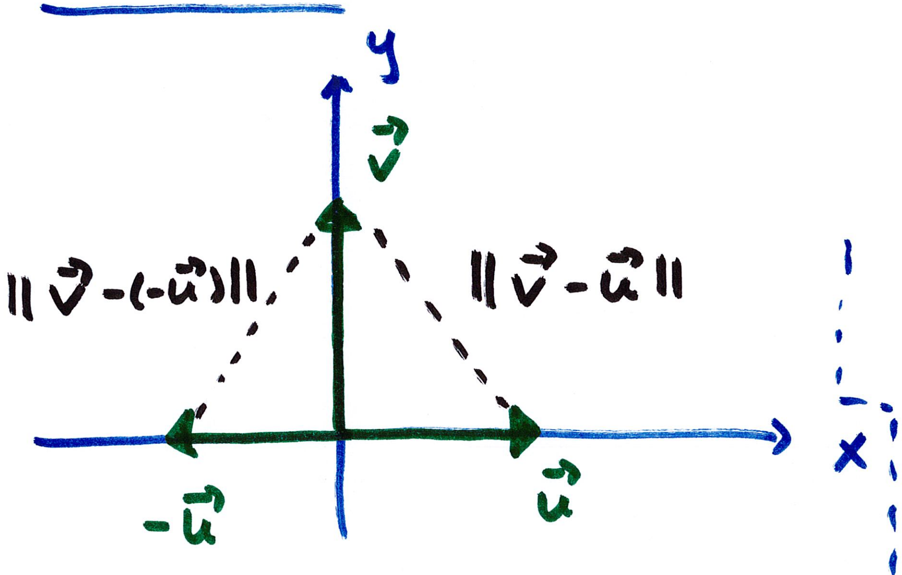


$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$$

$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \|\vec{v} - \vec{u}\|$$

$$\vec{u} - \vec{v} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \quad \|\vec{u} - \vec{v}\| = \sqrt{84}$$

orthogonality



$\vec{v}$ and $\vec{u}$ are orthogonal if

$$\|\vec{v} - \vec{u}\|^2 = \|\vec{v} + \vec{u}\|^2$$

$$(\vec{v} - \vec{u}) \cdot (\vec{v} - \vec{u}) = (\vec{v} + \vec{u}) \cdot (\vec{v} + \vec{u})$$

$$\cancel{\vec{v} \cdot \vec{v}} - \vec{v} \cdot \vec{u} - \vec{u} \cdot \vec{v} + \cancel{\vec{u} \cdot \vec{u}}$$

$$= \cancel{\vec{v} \cdot \vec{v}} + \vec{v} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \cancel{\vec{u} \cdot \vec{u}}$$

$$-2\,\vec{u} \cdot \vec{v} = 2\,\vec{u} \cdot \vec{v}$$

$$-\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{v}$$

true only if $\boxed{\vec{u} \cdot \vec{v} = 0}$

means

$$\vec{u} \perp \vec{v}$$

If W is a subspace of $\mathbb{R}^n$. Then all vectors orthogonal to every vector in W is called the <u>orthogonal complement</u> of W, written as $W^{\perp}$ (read: "W perpendicular" or "W perp")

$W^{\perp}$ is also a subspace of $\mathbb{R}^n$

<u>example</u> The z-axis is the orthogonal complement of the subspace that is the xy-plane.

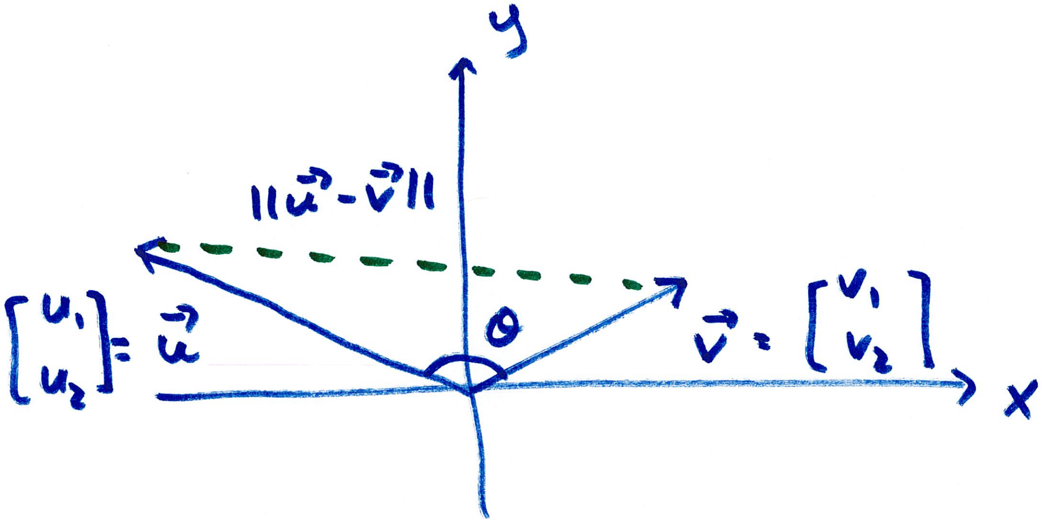
this is the same as $\underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{unit vector on} \\ \text{z-axis}}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\substack{\text{vectors} \\ \text{in } \mathbb{R}^3}} = 0$

look it as $A\vec{x} = \vec{0}$

this means the solution $\vec{x}$ is in the nullspace of A, $\dim \mathrm{Nul}(A) = 2$ (xy-plane)

$$\Rightarrow \boxed{(\mathrm{Row}\, A)^{\perp} = \mathrm{Nul}\, A}$$

likewise, $(\text{Col}\, A)^\perp = \text{Nul}\, A^T$



Law of cosines: $\|\vec{u}-\vec{v}\|^2 = \|\vec{v}\|^2 + \|\vec{u}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$

$$\|\vec{u}\|\|\vec{v}\|\cos\theta = \frac{-1}{2}\left(\|\vec{u}-\vec{v}\|^2 - \|\vec{v}\|^2 - \|\vec{u}\|^2\right)$$

$$= \frac{1}{2}\left(\|\vec{v}\|^2 + \|\vec{u}\|^2 - \|\vec{u}-\vec{v}\|^2\right)$$

$$= \frac{1}{2}\left(v_1^2 + v_2^2 + u_1^2 + u_2^2 - (u_1-v_1)^2 - (u_2-v_2)^2\right)$$

$$= \frac{1}{2}\left(\cancel{v_1^2} + \cancel{v_2^2} + \cancel{u_1^2} + \cancel{u_2^2} - \cancel{u_1^2} + 2u_1v_1 - \cancel{v_1^2}\right.$$
$$\left. - \cancel{u_2^2} + 2u_2v_2 - \cancel{v_2^2}\right)$$

$$\|\vec{u}\|\,\|\vec{v}\|\cos\theta = u_1v_1 + u_2v_2 = \vec{u}\cdot\vec{v}$$

so, $\boxed{\|\vec{u}\|\,\|\vec{v}\|\cos\theta = \vec{u}\cdot\vec{v}}$