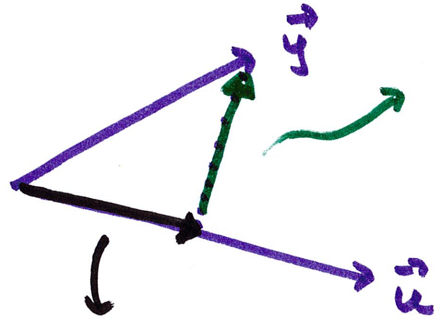
6.3 Orthogonal Projections

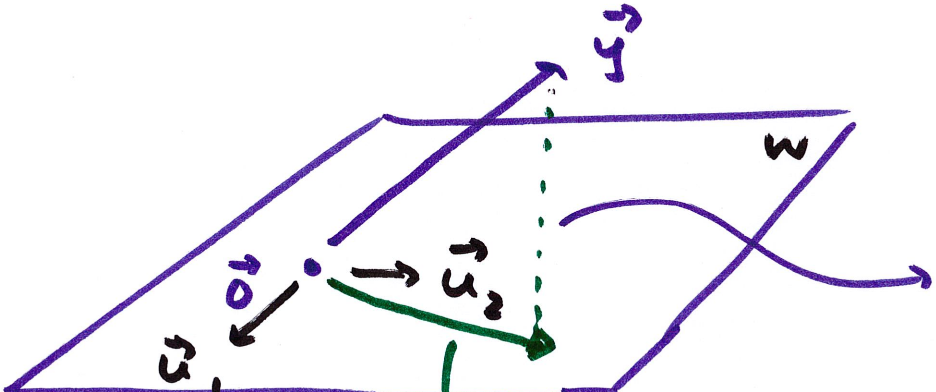
last time:



$\vec{z} = \vec{y} - \hat{y}$ component of $\vec{y}$ orthogonal to $\vec{u}$

$$\hat{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \mathrm{proj}_{\vec{u}}\,\vec{y}$$

orthogonal projection of $\vec{y}$ onto $\vec{u}$



$W = \text{span}\{\underbrace{\vec{u}_1, \vec{u}_2}_{\text{orthogonal}}\}$

$\vec{z} = \vec{y} - \hat{y}$

$$\hat{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2$$

if $\{\vec{u}_1, \dots, \vec{u}_p\}$ is orthogonal basis for W, then

projection of $\vec{y}$ onto W is uniquely determined as

$$\hat{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 + \cdots + \frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\vec{u}_p$$

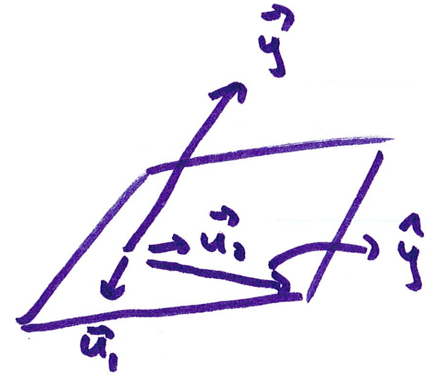
independent of the basis used (as long as it is orthogonal)

example $\vec{u}_1 = \begin{bmatrix} -7 \\ 1 \\ 4 \end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$ $\vec{y} = \begin{bmatrix} -9 \\ -1 \\ 1 \end{bmatrix}$

Find $\text{proj}_W \vec{y}$ $W = \text{span}\{\vec{u}_1, \vec{u}_2\}$

$\vec{u}_1 \cdot \vec{u}_2 = 0$ $\vec{u}_1 \perp \vec{u}_2$

$$\hat{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2$$



$$\hat{y} = \frac{63-1+4}{49+1+16}\begin{bmatrix}-7\\1\\4\end{bmatrix} + \frac{9-1-2}{1+1+4}\begin{bmatrix}-1\\1\\-2\end{bmatrix}$$

$$= (1)\begin{bmatrix}-7\\1\\4\end{bmatrix} + (1)\begin{bmatrix}-1\\1\\-2\end{bmatrix} = \begin{bmatrix}-8\\2\\2\end{bmatrix}$$

try another basis : $\left\{ \underset{\vec{u}_1}{\begin{bmatrix}1\\0\\-1\end{bmatrix}}, \underset{\vec{u}_2}{\begin{bmatrix}-3\\2\\-3\end{bmatrix}} \right\}$ also spans the same W

$$\vec{y} = \begin{bmatrix}-9\\-1\\1\end{bmatrix}$$

$$\hat{y} = \frac{-9-1}{1+1}\begin{bmatrix}1\\0\\-1\end{bmatrix} + \frac{27-2-3}{9+4+9}\begin{bmatrix}-3\\2\\-3\end{bmatrix}$$

$$= (-5)\begin{bmatrix}1\\0\\-1\end{bmatrix} + (1)\begin{bmatrix}-3\\2\\-3\end{bmatrix} = \begin{bmatrix}-8\\2\\2\end{bmatrix}$$ same as before

$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix}-9\\-1\\1\end{bmatrix} - \begin{bmatrix}-8\\2\\2\end{bmatrix} = \begin{bmatrix}-1\\-3\\-1\end{bmatrix}$$

the closest ~~the~~ distance between $\vec{y}$ and W is $\|\vec{z}\| = \sqrt{11}$

what is the point in W that is closest to $\vec{y}$?

$\rightarrow$ tip of $\hat{y}$: $(-8, 2, 2)$

$$\vec{u}_1 = \begin{bmatrix} -7 \\ 1 \\ 4 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} -9 \\ -1 \\ 1 \end{bmatrix}$$

find c_1 and c_2 such that $\vec{y} = c_1 \vec{u}_1 + c_2 \vec{u}_2$

but if $\vec{y}$ is not in span $\{\vec{u}_1, \vec{u}_2\}$, then there are no solutions.

$$\begin{bmatrix} -7 & -1 & -9 \\ 1 & 1 & -1 \\ 4 & -2 & 1 \end{bmatrix} \sim \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & \end{array} \\ \begin{bmatrix} 1 & 1 & -1 \\ 0 & 6 & -16 \\ 0 & 0 & \boxed{-11} \end{bmatrix} \end{array}$$

pivot in right most column

$\rightarrow$ inconsistent system

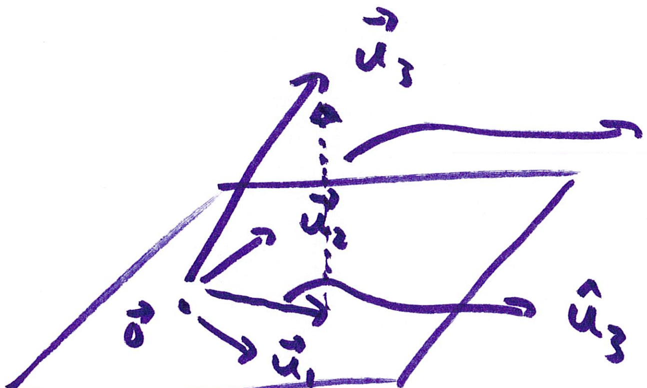
from earlier, $\hat{y} = (1)\vec{u}_1 + (1)\vec{u}_2$

the closest c_1, c_2 to the true solution.

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \qquad \vec{u}_2 = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} \qquad \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\vec{u}_1 \cdot \vec{u}_2 = 0$, so $\vec{u}_1 \perp \vec{u}_2$ but $\vec{u}_3$ is not orthogonal to either.

If we know $\vec{u}_3$ is not in span$\{\vec{u}_1, \vec{u}_2\}$, find a nonzero vector that is orthogonal to both $\vec{u}_1$ and $\vec{u}_2$.



$\vec{z}$ is $\perp$ to both $\vec{u}_1$ and $\vec{u}_2$

$$\hat{u}_3 = \frac{\vec{u}_3 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 + \frac{\vec{u}_3 \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}\vec{u}_2$$

$$= \frac{-2}{6}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \frac{2}{30}\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -2/5 \\ 4/5 \end{bmatrix}$$

$$\vec{z} = \vec{u}_3 - \hat{u}_3 = \begin{bmatrix} 0 \\ 2/5 \\ 1/5 \end{bmatrix}$$ this is orthogonal to both $\vec{u}_1$ and $\vec{u}_2$

If $\{\vec{u}_1, \dots, \vec{u}_p\}$ is an <u>orthonormal</u> basis for W

$\vec{u}_i \cdot \vec{u}_j = 0 \quad i \neq j$

$\|\vec{u}_i\| = 1$ for all i

$$\text{proj}_W \vec{y} = (\vec{y}\cdot\vec{u}_1)\vec{u}_1 + \dots + (\vec{y}\cdot\vec{u}_p)\vec{u}_p$$

if $U = [\vec{u}_1 \ \vec{u}_2 \ \cdots \ \vec{u}_p]$

then $\text{proj}_W \vec{y} = UU^T\vec{y}$

example $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix} -3 \\ 2 \\ -3 \end{bmatrix}$ $\vec{y} = \begin{bmatrix} -9 \\ -1 \\ 1 \end{bmatrix}$

$\{\vec{u}_1, \vec{u}_2\}$ is orthogonal but not orthonormal

but $\left\{ \frac{\vec{u}_1}{\|\vec{u}_1\|}, \frac{\vec{u}_2}{\|\vec{u}_2\|} \right\}$ is

$$= \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -3/\sqrt{22} \\ 2/\sqrt{22} \\ -3/\sqrt{22} \end{bmatrix} \right\} \quad U = \begin{bmatrix} 1/\sqrt{2} & -3/\sqrt{22} \\ 0 & 2/\sqrt{22} \\ -1/\sqrt{2} & -3/\sqrt{22} \end{bmatrix}$$

$$\hat{y} = \text{proj}_W \vec{y} = U U^T \vec{y} = \begin{bmatrix} 1/\sqrt{2} & -3/\sqrt{22} \\ 0 & 2/\sqrt{22} \\ -1/\sqrt{2} & -3/\sqrt{22} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ -3/\sqrt{22} & 2/\sqrt{22} & -3/\sqrt{22} \end{bmatrix} \begin{bmatrix} -9 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10/11 & -3/11 & -1/11 \\ -3/11 & 2/11 & -3/11 \\ -1/11 & -3/11 & 10/11 \end{bmatrix} \begin{bmatrix} -9 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \\ 2 \end{bmatrix}$$