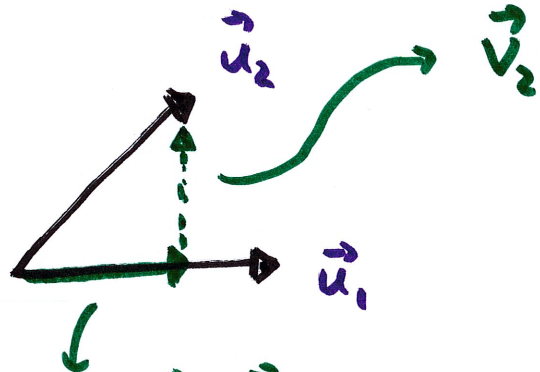
6.4 The Gram-Schmidt Process

$$\left\{ \underset{\vec{u}_1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}, \underset{\vec{u}_2}{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} \right\}$$

$\vec{u}_1$ and $\vec{u}_2$ are linearly independent but not orthogonal

modify this into an orthogonal set?



subtract from $\vec{u}_2$ the component along $\vec{u}_1$. Then $\vec{u}_1 = \vec{v}_1$ and $\vec{v}_2$ are orthogonal

$$\hat{u}_2 = \frac{\vec{u}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1$$

$$\cancel{\hat{u}_2 = \frac{11}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \qquad \hat{u}_2 = \frac{11}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11/5 \\ 22/5 \end{bmatrix}$$

$$\vec{v}_2 = \vec{u}_2 - \hat{u}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 11/5 \\ 22/5 \end{bmatrix} = \begin{bmatrix} 4/5 \\ -2/5 \end{bmatrix}$$

now $\{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4/5 \\ -2/5 \end{bmatrix} \right\}$ is orthogonal

<u>example</u>

$$\left\{ \underset{\vec{x}_1}{\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}}, \underset{\vec{x}_2}{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{x}_3}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \right\}$$

not mutually orthogonal

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - \underbrace{\frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}}_{\substack{\text{component} \\ \text{of } \vec{x}_2 \\ \text{along } \vec{v}_1}} \vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} - \frac{-3}{6} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3/2 \\ -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$$

make fractions disappear (for convenience)

$$\vec{v}_3 = \vec{x}_3 - \underbrace{\frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}}_{\substack{\text{gets rid of} \\ \text{component of} \\ \vec{x}_3 \,/\!/\, \vec{v}_1}} \vec{v}_1 - \underbrace{\frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}}_{\substack{\text{gets rid} \\ \text{of component} \\ \text{of } \vec{x}_3 \,/\!/\, \vec{v}_2}} \vec{v}_2$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{14} \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 17/21 \\ 34/21 \\ 68/21 \end{bmatrix} = \begin{bmatrix} 17 \\ 34 \\ 68 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

now $\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \right\}$ is orthogonal

in general, given $\{ \vec{x}_1, \ldots, \vec{x}_p \}$, Gram-Schmidt process is

$$\vec{v}_1 = \vec{x}_1$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_2 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$\vec{v}_4 = \vec{x}_4 - \frac{\vec{x}_4 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_4 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 - \frac{\vec{x}_4 \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3} \vec{v}_3$$

$$\vdots$$

QR factoritation of matrices:

If A is an $m \times n$ matrix with linearly independent columns, then $A = QR$ where Q is an $m \times n$ matrix whose columns form an <u>orthonormal</u> basis for Col A and R is an $n \times n$ ~~matrix~~ upper triangular matrix w/ positive entries on the main diagonal.

<u>example</u>

$$A = \begin{array}{cc} \begin{bmatrix} 0 & 4 \\ 8 & 6 \\ 4 & -7 \end{bmatrix} \\ \quad \vec{x}_1 \quad \vec{x}_2 \end{array}$$

independent but not orthogonal

first, apply Gram-Schmidt process

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix} - \frac{20}{80} \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -8 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

are not ~~nor~~ orthonormal yet

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} 0 \\ 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \qquad \vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 1/\sqrt{6} \\ 2/\sqrt{5} & 1/\sqrt{6} \\ 1/\sqrt{5} & -2/\sqrt{6} \end{bmatrix} \qquad A = QR$$

$$\underset{3\times 2}{\begin{bmatrix} 0 & 4 \\ 8 & 6 \\ 4 & -7 \end{bmatrix}} = \underset{3\times 2}{\begin{bmatrix} 0 & 1/\sqrt{6} \\ 2/\sqrt{5} & 1/\sqrt{6} \\ 1/\sqrt{5} & -2/\sqrt{6} \end{bmatrix}} \underset{2\times 2}{R}$$

$$Q = \begin{bmatrix} 0 & 1/\sqrt{6} \\ 2/\sqrt{5} & 1/\sqrt{6} \\ 1/\sqrt{5} & -2/\sqrt{6} \end{bmatrix} \qquad Q^T = \begin{bmatrix} 0 & 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix}$$

$$Q^T Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A = QR$$

$$Q^T A = Q^T Q R = R$$

$$R = \begin{bmatrix} 0 & 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 8 & 6 \\ 4 & -7 \end{bmatrix} = \begin{bmatrix} 20/\sqrt{5} & 1/\sqrt{5} \\ 0 & 24/\sqrt{6} \end{bmatrix}$$