6.7 Inner Product Spaces

For $\vec{u}, \vec{v}$ in $\mathbb{R}^n$, the standard inner product (dot product)

is $\langle \vec{u}, \vec{v} \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$

4 properties of inner product

1) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

2) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

3) $\langle c\vec{u}, \vec{v} \rangle = c \langle \vec{u}, \vec{v} \rangle$

4) $\langle \vec{u}, \vec{u} \rangle \geq 0$ and $\langle \vec{u}, \vec{u} \rangle = 0$ if and only if $\vec{u} = \vec{0}$

any vector space with ~~the~~ inner product according to the 4 properties above is an <u>inner product space</u>.

But we can define inner product in other ways.

e.g. $\langle \vec{u}, \vec{v} \rangle = 3u_1 v_1 + 4u_2 v_2$ in $\mathbb{R}^2$ — this satisfies the 4 axioms above if inner product changes, then "length" and "orthogonality" also change their means.

for example, if $\langle \vec{u}, \vec{v} \rangle = 3u_1v_1 + 4u_2v_2$

$$\vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{(3)(3)(3) + (4)(1)(1)} = \sqrt{31}$$

$\sqrt{31}$ is the "length" or $\vec{x}$ in this inner product space

find $\vec{z}$ such that $\vec{z}$ is "orthogonal" to $\vec{y}$?

$\vec{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ want $\langle \vec{z}, \vec{y} \rangle = 0$

$$\langle \vec{z}, \vec{y} \rangle = 3(z_1)(5) + 4(z_2)(-1) = 0$$

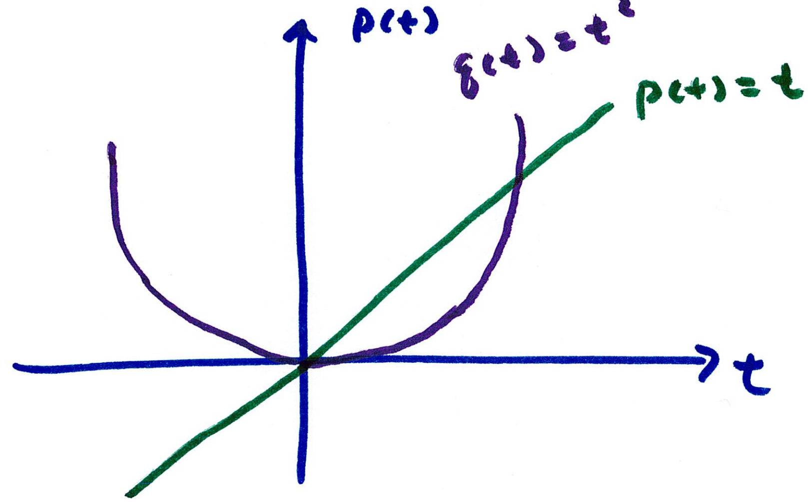
$$z_1 = \frac{4}{15} z_2$$

$$\vec{z} = \begin{bmatrix} \frac{4}{15} z_2 \\ z_2 \end{bmatrix} = z_2 \begin{bmatrix} 4/15 \\ 1 \end{bmatrix}$$

any vector ~~parral~~ parallel to $\begin{bmatrix} 4 \\ 15 \end{bmatrix}$ is $\perp$ to $\vec{y}$

Distance, length, and orthogonality of polynomials?

$p(t) = t \qquad q(t) = t^2$



what does $\langle p, q \rangle$ mean?

we need to define the inner product.

one way is to base it on values of p and q at several t, for example, $t = -1, 0, 1$

then define $\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$

$\langle p, p \rangle = \|p\|^2 = p(-1)p(-1) + p(0)p(0) + p(1)p(1)$

$= (-1)(-1) + (0)(0) + (1)(1) = 2$

so "length" or $p(t)$ is $\|p\| = \sqrt{2}$

$$\langle g, g\rangle = \|g\|^2 = g(-1)\,g(-1) + g(0)\,g(0) + g(1)\,g(1)$$

$$= 1 + 0 + 1 = 2 \qquad \|g\| = \sqrt{2}$$

$$\langle p, g\rangle = p(-1)\,g(-1) + p(0)\,g(0) + p(1)\,g(1)$$

$$= (-1)(1) + (0)(0) + (1)(1) = 0$$

so $p(t) = t$ and $g(t) = t^2$ are orthogonal

the set $\{1, \underbrace{t, t^2}_{\text{are orthogonal (from work above)}}\}$ is ~~orthogonal~~ linearly independent, but is it orthogonal?

$p_1(t) = 1$, $p_2(t) = t$, $p_3(t) = t^2$

$$\langle p_1, p_2\rangle = p_1(-1)\,p_2(-1) + p_1(0)\,p_2(0) + p_1(1)\,p_2(1)$$

$$= (1)(-1) + (1)(0) + (1)(1) = 0 \qquad \text{so } p_1 \perp p_2$$

$$\langle p_1, p_3\rangle = p_1(-1)\,p_3(-1) + p_1(0)\,p_3(0) + p_1(1)\,p_3(1)$$

$$= (1)(1) + (1)(0) + (1)(1) = 2$$

$\{1, t, t^2\}$ is not orthogonal, but we can apply Gram-Schmidt to find an orthogonal set.

$p_1(t) = 1, \quad p_2(t) = t, \quad p_3(t) = t^2$

$\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$

$q_1(t) = p_1(t) = 1$

$q_2(t) = p_2(t) = t$ because we know, from last page, $p_1 \perp p_2$

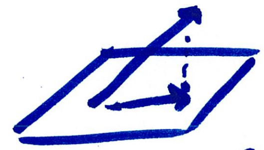
$$q_3(t) = p_3(t) - \frac{\langle p_3, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1 - \frac{\langle p_3, q_2 \rangle}{\langle q_2, q_2 \rangle} q_2$$

$$= t^2 - \frac{2}{3}(1) - \frac{0}{2} t = t^2 - \frac{2}{3} = 3t^2 - 2$$

so $\{1, t, 3t^2 - 2\}$ is an orthogonal set and an orthogonal basis for $\mathbb{P}_2$

what is the best approximation of t^3 in $\mathbb{P}_2$?

$\rightarrow$ the "shadow" of t^3 in $\mathbb{P}_2$

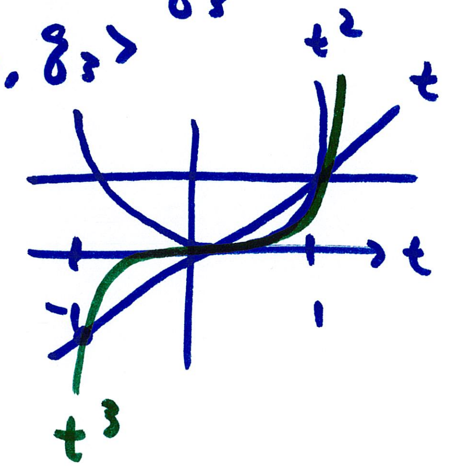


let $r(t) = t^3$, find $\hat{r}(t) = \text{proj}_{\mathbb{P}_2} t^3$

$$\{\underset{g_1}{\underset{\uparrow}{1}}, \underset{g_2}{\underset{\uparrow}{t^2}}, \underset{g_3}{\underset{\uparrow}{3t^2-2}}\}$$

$$\hat{r}(t) = \frac{\langle r, g_1 \rangle}{\langle g_1, g_1 \rangle} g_1 + \frac{\langle r, g_2 \rangle}{\langle g_2, g_2 \rangle} g_2 + \frac{\langle r, g_3 \rangle}{\langle g_3, g_3 \rangle} g_3$$

$= \cdots = t$ (because t and t^3 are closest at $\underbrace{t = -1, 0, 1}_{\text{why these?}}$)



why not sample ALL t between $t=a$, $t=b$?

so, we can do "standard" inner product

$$\langle f, g \rangle = \int_a^b f(t)g(t)\,dt$$

for f, g in $C[a,b]$ $\rightarrow$ all continuous functions on $a \le t \le b$

<u>example</u> $f(t) = \cos t \quad g(t) = \sin t \quad -\pi \le t \le \pi$

$$\langle f, g \rangle = \int_{-\pi}^{\pi} \cos t \sin t \, dt = \frac{1}{2} \sin^2 t \Big|_{-\pi}^{\pi} = 0$$

so $\sin t$ and $\cos t$ are orthogonal

so are $\cos(nt)$ and $\cos(mt)$ when $n \neq m$

(and $\sin(nt)$ and $\sin(mt)$)

using $\langle f, g \rangle = \int_a^b f(t) g(t) \, dt$, the best approximation of t^3 by a function in $\mathbb{P}_2$ on $-1 \le t \le 1$

is $\frac{3}{5} t$