7.1 Diagonalization of Symmetric Matrices

Symmetric matrix : matrix A such that $A^T = A$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

try diagonalizing $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} = PDP^{-1}$

$$\det(A - \lambda I) = 0 \qquad \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 - 9 = 0 \qquad 1-\lambda = 3 \quad \text{or} \quad 1-\lambda = -3$$

$$\lambda = -2 \,, \quad \lambda = 4$$

eigenvectors: $\lambda = -2$ solve $(A-\lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$\lambda = 4$ solve $(A-\lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

note the eigenvectors from different eigenvalues are <u>orthogonal</u> → ALWAYS true for symmetric matrices

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

the columns of P are <u>orthonormal</u>

when a square matrix has orthonormal columns, it is an <u>orthogonal matrix</u>

$$P^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$P^TP = PP^T = I$ → if matrix is orthogonal, its transpose = its inverse

$$A = PDP^{-1} = PDP^T$$

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

if a matrix can be diagonalized such that P is orthogonal, then we say the matrix is <u>orthogonally diagonalizable</u>

if $A = A^T$, then it is ~~dia~~ orthogonally diagonalizable

but is the reverse true? (if orthogonally diagonalizable, ~~did~~ is it always symmetric?)

if $A = PDP^T$ is $A = A^T$?

$$A^T = (PDP^T)^T = (P^T)^T D^T P^T$$

$$= PDP^T = A \Rightarrow A^T = A$$, so yes.

$$\boxed{\text{symmetric} \leftrightarrow \text{orthogonally diagonalizable}}$$

repeated eigenvalues

<u>example</u>

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\lambda = -1, -1, 2$$

eigenvector for $\lambda = 2$ ... $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

$\lambda = -1$: $(A - \lambda I)\vec{v} = \vec{0}$

from last page, we know if A is symmetric, then it is orthogonally diagonalizable, which means the dimension of the eigenspace must match the multiplicity of the corresponding eigenvalue.

so, here, $\lambda = -1$ twice, there is guaranteed to be TWO eigenvectors for $\lambda = -1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_2, x_3 free

$x_1 = -x_2 - x_3$

$$\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$\vec{v}_1$ is orthogonal to $\vec{v}_2$ and $\vec{v}_3$, ~~be~~ and this is ALWAYS true because $\vec{v}_1$ and $\{\vec{v}_2, \vec{v}_3\}$ are from distinct eigenvalues.

P must have orthonormal columns. But $\vec{v}_2$ is not orthogonal to $\vec{v}_3$

Perform Gram-Schmidt process to change $\vec{v}_3$

$$\vec{u}_3 = \vec{v}_3 - \frac{\langle \vec{v}_3, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \vec{v}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \|\vec{u}_3\| = \frac{\sqrt{6}}{2\sqrt{2}}$$

$$\vec{w}_3 = \frac{\vec{u}_3}{\|\vec{u}_3\|} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

so now $\{\vec{v}_1, \vec{v}_2, \vec{w}_3\}$ is orthonormal and form columns of P

$$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

P is orthogonal, $P^T = P^{-1}$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$