"Homework 4" and "Homework 5" are due Tomorrow

1.4 The Matrix Equation $A\vec{x} = \vec{b}$

$$x_1 + 2x_2 + 3x_3 = 4$$
$$5x_1 + 6x_2 + 7x_3 = 8$$
$$9x_1 + 10x_2 + 11x_3 = 12$$

can be written as $\underbrace{x_1 \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}}_{\text{vector equation}}$

this can be expressed as a <u>matrix equation</u>

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}}_{\vec{b}}$$

in general, if $\vec{a}_1, \vec{a}_2, \vec{a}_3, \dots, \vec{a}_n$ are columns of A and $x_1, x_2, \dots, x_n$ are elements of column vector $\vec{x}$

then
$$\underbrace{\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n$$

this gives us a way to multiply a matrix by a vector

<u>example</u>
$$\begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 6 \\ -4 \\ 7 \end{bmatrix} + 3\begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 27 \\ -17 \\ 32 \end{bmatrix}$$

$3 \times \boxed{2}$ $\boxed{2} \times 1$

these MUST match

"row-vector rule"

multiply first ~~column~~ row of A by first column of $\vec{x}$

then 2nd row A by first col. of $\vec{x}$

and so on

$$= \begin{bmatrix} 6\cdot 2 + 5\cdot 3 \\ -4\cdot 2 + -3\cdot 3 \\ 7\cdot 2 + 6\cdot 3 \end{bmatrix} = \begin{bmatrix} 27 \\ -17 \\ 32 \end{bmatrix}$$

each element is a sum of products

example $\begin{bmatrix} 7 & -3 \\ 2 & 1 \\ 9 & -6 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ 12 \\ -4 \end{bmatrix}$

$\underset{=}{4} \times \boxed{2}$ $\quad$ $\boxed{2} \times \underset{=}{1}$ $\quad$ 4×1

equation $A\vec{x} = \vec{b}$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ -2 & -4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 9 \end{bmatrix}$$

solution: $\begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ -2 & -4 & -3 & 9 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

so $x_1 = 0$, $x_2 = -3$, $x_3 = 1$

$$\vec{x} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$

so this means $0 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + (-3) \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} + (1) \begin{bmatrix} 4 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 9 \end{bmatrix}$

therefore, for a solution $\vec{x}$ to exist, the vector $\vec{b}$

MUST be a linear combination of columns of A.

$\Rightarrow$ $\vec{b}$ must be in span$\{\vec{a}_1\ \vec{a}_2\ \vec{a}_3\}$

if we know $\vec{b}$, we know how to find $\vec{x}$

question: for an A that is 3×3, can we form every possible vector in $\mathbb{R}^3$

in other words, is span$\{\vec{a}_1\ \vec{a}_2\ \vec{a}_3\} = \mathbb{R}^3$

suppose $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ -2 & -4 & -3 \end{bmatrix}$

is there a solution $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

such that $x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

for ANY b_1, b_2, b_3

$$\begin{bmatrix} 1 & 2 & 4 & b_1 \\ 0 & 1 & 5 & b_2 \\ -2 & -4 & -3 & b_3 \end{bmatrix}$$

$$\sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & \frac{17}{5} b_1 - 2b_2 + \frac{6}{5} b_3 \\ 0 & 1 & 0 & b_2 - 2b_1 - b_3 \\ 0 & 0 & 1 & \frac{2}{5} b_1 + \frac{1}{5} b_3 \end{bmatrix}$$

there is a pivot in each column, so a solution <u>always exists</u> for ANY b_1, b_2, b_3

therefore, this matrix's columns span $\mathbb{R}^3$

what if we got $\begin{bmatrix} 1 & 0 & 0 & b_1 - 2b_2 + b_3 \\ 0 & 1 & 0 & b_2 - 3b_1 + b_3 \\ 0 & 0 & 0 & b_1 + b_2 + b_3 \end{bmatrix}$

so if $b_1 + b_2 + b_3 \neq 0$, then there is no solution

so $\text{span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\} \neq \mathbb{R}^3$

the columns of an $m \times n$ matrix A span $\mathbb{R}^m$ if the reduced row echelon form of A has a pivot position in every column

then, the follow four statements are logically equivalent

a. For each $\vec{b}$ in $\mathbb{R}^m$, $A\vec{x} = \vec{b}$ has a solution

b. Each $\vec{b}$ in $\mathbb{R}^m$ is a linear combo of columns of A

c. Columns of A span $\mathbb{R}^m$

d. A has pivot position in every row

1.5 Solution sets of Linear Systems

$A\vec{x} = \vec{b}$ is called <u>homogeneous</u> if $\vec{b} = \vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$A\vec{x} = \vec{0}$ <u>always</u> has <u>the trivial solution</u> $\vec{x} = \vec{0}$

$A\vec{x} = \vec{0}$ has nontrivial solutions if A does not have pivot position in every row

<u>example</u> $A = \begin{bmatrix} 3 & 3 & 6 \\ -9 & -9 & -18 \\ 0 & -4 & 12 \end{bmatrix}$

$$A\vec{x} = \vec{0} \rightarrow \begin{bmatrix} 3 & 3 & 6 & 0 \\ -9 & -9 & -18 & 0 \\ 0 & -4 & 12 & 0 \end{bmatrix}$$

$$\sim \cdots \sim \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

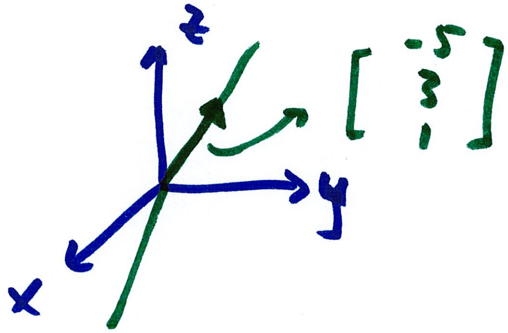
x_3 free

$x_2 = 3x_3$

$x_1 = \cancel{-3x} - x_2 - 2x_3 = -3x_3 - 2x_3 = -5x_3$

so $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = \underbrace{x_3 \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}}$

line in $\mathbb{R}^3$



parametric vector form
(x_3 is the parameter)

nonhomogeneous system

same A : $A = \begin{bmatrix} 3 & 3 & 6 \\ -9 & -9 & -18 \\ 0 & -4 & 12 \end{bmatrix}$

$\vec{b} = \begin{bmatrix} 12 \\ -36 \\ 8 \end{bmatrix}$

Solve: $\begin{bmatrix} 3 & 3 & 6 & 12 \\ -9 & -9 & -18 & -36 \\ 0 & -4 & 12 & 8 \end{bmatrix}$

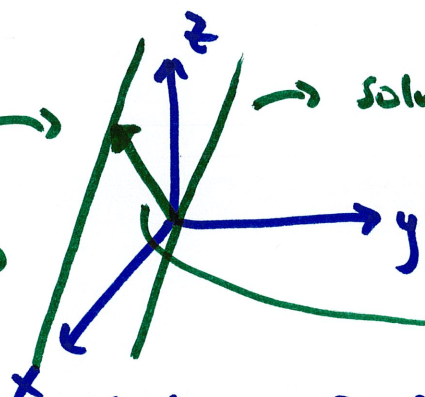
$\sim \cdots \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

x_3 free

$x_2 = -2 + 3x_3$

$x_1 = -x_2 - 2x_3 + 4 = 6 - 5x_3$

solution: $\vec{x} = \begin{bmatrix} 6-5x_3 \\ -2+3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 0 \end{bmatrix} + \underbrace{x_3 \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}}_{\text{solution of } A\vec{x}=\vec{0}}$



solution of $\vec{b} \neq 0$

solution of $A\vec{x} = \vec{0}$

same slopes

solution of $A\vec{x} = \vec{b}$ is ~~sol~~ a shifted version of solution of $A\vec{x} = \vec{0}$ if $A\vec{x} = \vec{b}$ is consistent