"HW 6" and "HW 7" are due together

1.7 Linear Independence

a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is <u>linearly independent</u>

if $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}$

can happen if and only if $x_1 = x_2 = x_3 = \cdots = x_n = 0$

example: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

true if $x_1 = x_2 = 0$

so this set of vectors is linearly independent

this means if $A\vec{x} = \vec{0}$ has only the trivial solution

then the columns of A are linearly independent a set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ are <u>linearly dependent</u>

if the set is not linearly independent

example: $x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix} + x_3\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

<u>in addition</u> to $x_1 = x_2 = x_3 = 0$

a $x_1 = 1$, $x_2 = 1$, $x_3 = -1$ is also a solution

so $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$ is linearly <u>dependent</u>

if $A = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\end{bmatrix}$

then $A\vec{x} = \vec{0}$ has nontrivial solution

$$\begin{array}{c}\left[\begin{array}{cccc}1 & 0 & 1 & 0\\0 & 1 & 1 & 0\end{array}\right]\\ \begin{array}{cccc}\vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \end{array}\end{array}$$

two pivots, three variables

x_3 free

$x_2 = -x_3$

$x_1 = -x_3$

one possible solution: $x_1 = -1, \quad x_2 = -1, \quad x_3 = 1$

$$\Rightarrow (-1)\vec{a}_1 + (-1)\vec{a}_2 + (1)\vec{a}_3 = \vec{0} \rightarrow \text{linear dependence relation}$$

Sometime we can tell if vectors are dependent by inspection

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \end{bmatrix} \right\} \qquad (-2)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$

one is multiple of another $\Rightarrow$ dependent

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$$

not multiples of one another

can check if ~~indep~~ independent by solving $A\vec{x} = \vec{0}$

or, $x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

<u>if</u> $x_2 \neq 0$, then $\begin{bmatrix} 1 \\ 3 \end{bmatrix} = -\frac{x_1}{x_2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

but if this true, then the vectors are multiples of one another, which is FALSE. So, x_2 must be zero.

which means $x_1 = 0$, so independent.

if a set contains $\vec{0}$, it MUST be dependent set

$$\{\vec{a_1}, \vec{a_2}, \vec{0}\}$$

$x_1 \vec{a_1} + x_2 \vec{a_2} + x_3 \vec{0} = \vec{0}$ means $x_1 = x_2 = x_3 = 0$
is the only solution
$\Rightarrow$ independent

$x_1 = 0$, $x_2 = 0$, $x_3 = 1$

nontrivial so set $\underline{\underline{\text{is}}}$ dependent.

if set is dependent, not <u>every</u> vector has to be

a linear combo of the others.

<u>example</u> : $\vec{u} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$ $\vec{z} = \begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix}$

independent? $\underbrace{\begin{bmatrix} \vec{u} & \vec{v} & \vec{w} & \vec{z} \end{bmatrix}}_{A} \vec{x} = \vec{0}$

$$\begin{bmatrix} 3 & -6 & 0 & 3 & 0 \\ 2 & 1 & -5 & 7 & 0 \\ -4 & 7 & 2 & -5 & 0 \end{bmatrix}$$

$$\sim \cdots \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

x_3 free

$x_4 \quad 0$

$x_2 = -x_3$

$x_1 = -3x_3$

so set is <u>dependent</u>

dependence relation:

let $x_3 = 1$, $x_1 = -3$, $x_2 = -1$, $x_4 = 0$

so $(-3)\vec{u} + (-1)\vec{v} + (1)\vec{w} = \vec{0}$

$\vec{z}$ is not linear combo of others.

if there are more than n vectors in a set of $n \times 1$ vectors then the set is dependent

why? if $n=3$

then $\underbrace{\text{rref}}_{\substack{\text{reduced} \\ \text{row} \\ \text{echelon} \\ \text{form}}}$ of $A\vec{x} = \vec{0}$

$\downarrow$

columns are the vectors in the set

is either $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

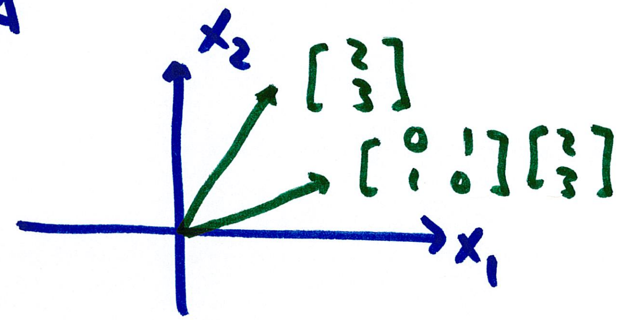
or $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

1.8 Introduction to Linear Tranformations

$A\vec{x} = \vec{b}$ is a system

but we can also look at it as a transformation of $\vec{x}$ into $\vec{b}$ by the matrix A

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



this ~~or~~ transformation flips x_1, x_2 coords of 2D vector

notation: $T(\vec{x}) = A\vec{x}$ $\qquad \vec{x} \mapsto A\vec{x}$

$\vec{x}$: input domain of tranformation T

$A\vec{x}$: out range or codomain of T

an $m \times n$ matrix A transforms an n-vector into an m-vector $\quad T: \mathbb{R}^n \to \mathbb{R}^m$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \qquad A\vec{x} \text{ means } \vec{x} \text{ is } 2 \times 1$$

3×2

requirement

for simplicity

$$\underset{3\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} -1 \\ -5 \end{bmatrix}} = \underbrace{\underset{3\times 1}{\begin{bmatrix} -11 \\ -23 \\ -5 \end{bmatrix}}}_{\text{"image" of } T} \qquad T: \mathbb{R}^2 \to \mathbb{R}^3$$

if $A = \begin{bmatrix} 1 & -7 & -26 \\ -4 & 22 & 80 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$

$T : \mathbb{R}^3 \to \mathbb{R}^2$

How many vectors ~~can~~ in $\mathbb{R}^3$ can be transformed into $\vec{b}$?

$T(\vec{x}) = A\vec{x}$ $A\vec{x} = \vec{b}$ $\vec{x} = ?$

this is just a system!

$$\begin{bmatrix} 1 & -7 & -26 & -3 \\ -4 & 22 & 80 & 6 \end{bmatrix}$$

$$\sim \cdots \sim \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 4 & 1 \end{bmatrix}$$

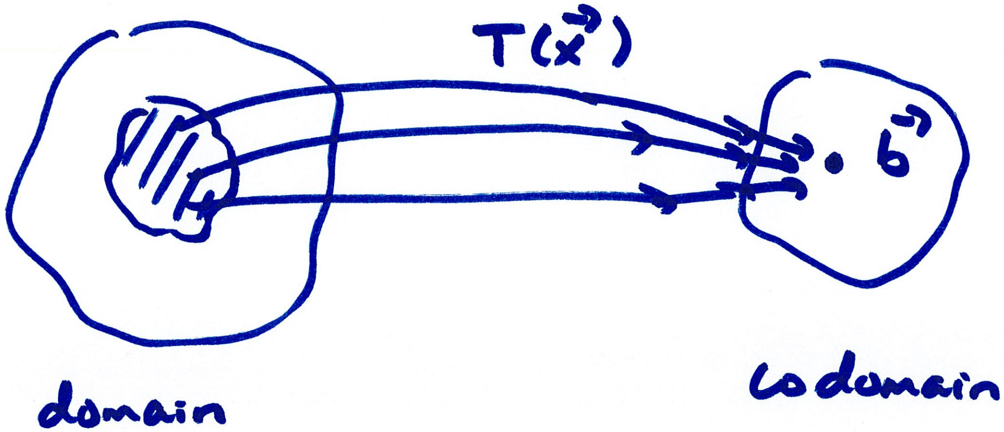
x_3 is free

$x_2 = 1 - 4x_3$

$x_1 = 4 - 2x_3$

$$\vec{x} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -4 \\ 1 \end{bmatrix}$$

infinitely many $\vec{x}$ turn into $\vec{b}$



Basic Properties of Linear Transformation

$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

$A\begin{bmatrix}3\\1\\3\end{bmatrix} = A\begin{bmatrix}1\\0\\1\end{bmatrix} + A\begin{bmatrix}2\\1\\2\end{bmatrix}$

$\square \times 3 \quad 3\times 1$

$T(c\vec{u}) = cT(\vec{u})$

$T(\vec{0}) = \vec{0}$ → an easy way to test if a transformation is linear

$A\vec{x}$ is ALWAYS linear

$T(c\vec{u}+d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$