2.1 Matrix Operations

Suppose A is an $n \times m$ matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & \cdots & a_{1m} \\ 4 & 5 & 6 & \cdots & \cdot \\ \vdots & \ddots & & & \\ \cdot & \cdots & \cdot & \cdots & a_{nm} \end{bmatrix}$$

$\vec{a}_1$ $\quad$ $\vec{a}_3$ $\quad$ a_{nm}

lower case of matrix name is used to denote individual elements

$a_{11} = 1$

row $\quad$ column

$a_{23} = 6$

a <u>diagonal matrix</u> is a square matrix has zeros everywhere except possibly on the <u>main diagonal</u>

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

a **zero matrix** has zeros everywhere

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad 0 \rightarrow \text{big zero}$$

a **identity matrix** has 1's on the main diagonal $\boxed{\text{square}}$

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & 0 & 1 & \ddots \\ \vdots & \vdots & \ddots & 1 \end{bmatrix} \qquad I \quad \text{~~or~~ e.g.} \quad I_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

two matrices are *equal* if they have the same size *and* the corresponding elements are the same

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

addition and subtraction are done with matrices of the <u>same size</u>

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not defined

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

$$= \underbrace{(-4)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{\text{scalar multiple of matrix } \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}$$

$$(2)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (3)\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 16 \end{bmatrix}$$

Matrix multiplication:

we already know how to do
$$\begin{bmatrix} \boxed{1 \quad 2} \\ 3 \quad 4 \\ 5 \quad 6 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} \boxed{-5} \\ -11 \\ -17 \end{bmatrix}$$

$3 \times \boxed{2}$ $\quad \boxed{2} \times 1$ $\quad 3 \times 1$

$=$ $\quad =$

must match

very similar if multiplying by a matrix

$$\begin{bmatrix} \boxed{1 \quad 2} \\ 3 \quad 4 \\ 5 \quad 6 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} \boxed{-1} & 6 \\ -3 & 10 \\ -5 & 14 \end{bmatrix}$$

$3 \times \boxed{2}$ $\quad \boxed{2} \times 2$

$=$ $\quad =$

must match

rows of first matrix multiplying each row of first column of second matrix

Same thing but w/ 2nd column of 2nd matrix

"row-column rule"

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 0 & 4 \end{bmatrix}$$ is also

$$= \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix}$$

where $\vec{b}_1$ is linear combo of columns of first matrix using elements of first of second matrix as weights

similar idea for $\vec{b}_2$ (2nd col. of 2nd matrix as weights)

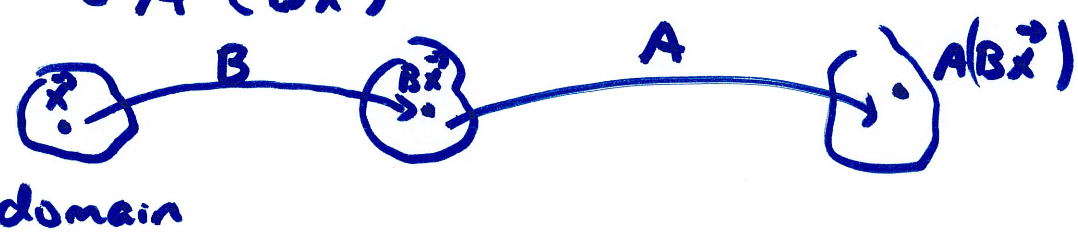
so, AB, if defined, and if $B = [\vec{b}_1\ \vec{b}_2\ \vec{b}_3\ \dots]$

then $AB = [A\vec{b}_1\ \ A\vec{b}_2\ \ A\vec{b}_3\ \ \dots]$

AB is also two linear mappings consecutively

order is VERY important

$$AB\vec{x} = A(B\vec{x})$$



$\boxed{\text{NOTE: in general, } AB \neq BA}$ even if both are defined

$$A = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 11 & 10 \\ -10 & -11 \end{bmatrix} \qquad BA = \begin{bmatrix} -5 & -2 \\ -2 & 5 \end{bmatrix}$$

also, we <u>cannot</u> "cancel" like with numbers

$$(3)(\cancel{2}) = (x)(\cancel{2}) \rightarrow x = 3$$

$$A\cancel{B}\vec{x} = \cancel{B}C \quad \textbf{NO!}$$

matrix power: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad A^2 \neq \begin{bmatrix} 1^2 & 2^2 \\ 3^2 & 4^2 \end{bmatrix}$

instead, $A^2 = AA$ (so A must be square)

e.g. $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$B^2 = \underset{2\times 1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$ is NOT defined

$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$A^3 = AAA$

$A^4 = AAAA$ and so on

A^T means the <u>transpose</u> of A

$\rightarrow$ swap rows and columns

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$$

properties:

$$(A^T)^T = A$$

$$(A+B)^T = A^T + B^T$$

$$(cA)^T = cA^T$$

$$(AB)^T = B^T A^T$$

why $(AB)^T = B^T A^T$?

let A be 2×3 and B be 3×5

$\underset{2\times 3}{A}\,\underset{3\times 5}{B}$ is 2×5

$\underset{3\times 2}{A^T}\,\underset{5\times 3}{B^T}$ ~~is~~ is not defined

but $\underset{5\times 3}{B^T}\,\underset{3\times 2}{A^T}$ is defined (5×2)