2.2 + 2.3 The Inverse of a Matrix

scalar inverse : $5^{-1} \cdot 5 = 1$ and $5 \cdot 5^{-1} = 1$

5^{-1} is the inverse of the scalar 5

matrix inverse : $A^{-1}A = I$ and $AA^{-1} = I \rightarrow$ identity $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A^{-1} is the inverse of matrix A

A ~~is a~~ must be square

NOT every square matrix has an inverse

if A has an inverse, A is <u>invertible</u>

if A does not, A is <u>singular</u>

(invertible matrix is often called nonsingular)

2 x 2 Case

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

if $ad - bc \neq 0$ then $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

if $\underbrace{ad - bc}_{\text{determinant of } A} = 0$, then A is <u>NOT</u> invertible

example $A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ determinant: $(4)(1) - (3)(2) = -2 \neq 0$

so A^{-1} exists

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -1/2 & 3/2 \\ +1 & -2 \end{bmatrix}$$

check: $AA^{-1} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1/2 & 3/2 \\ +1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$$A^{-1}A = \begin{bmatrix} -1/2 & 3/2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

if A^{-1} exists, then $A\vec{x} = \vec{b}$ has a <u>unique</u> solution

because $\underbrace{A^{-1}A}_{I}\vec{x} = A^{-1}\vec{b}$ pre-multiplying by A^{-1}

ORDER IS IMPORTANT

$$\vec{x} = A^{-1}\vec{b}$$

<u>example</u>

$$8x_1 + 6x_2 = 2$$
$$5x_1 + 4x_2 = -1$$

$$\underbrace{\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{\vec{b}}$$

$$A^{-1} = \frac{1}{32-30} \begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}$$

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -9 \end{bmatrix}$$

if A^{-1} does not exist, then $A\vec{x} = \vec{b}$ can have no solution or infinitely-many solution

if A^{-1} exists, then $\vec{b}$ is a linear combo of columns of A.

A more general method to find A^{-1} (any $n \times n$)

Form an augmented matrix $[A \ \ I]$

then row reduce left half to I, the resulting right half is A^{-1}

(if the left half cannot be reduced to I, then A^{-1} does not exist)

$$[A \ \ I] \sim \cdots \sim [I \ \ A^{-1}]$$

<u>example</u>

$$A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc:cc} 4 & 3 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc:cc} 2 & 1 & 0 & 1 \\ 4 & 3 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc:cc} 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{cc:cc} 2 & 0 & -1 & 3 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc} 1 & 0 & -1/2 & 3/2 \\ 0 & 1 & 1 & -2 \end{array}\right] \rightarrow A^{-1}$$

example

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

<u>example</u>

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & -7 & -2 & 1 \end{bmatrix}$$

no way this can turn into I

so A is <u>singular</u>

the above examples indicate that if A is row-equivalent to I then A^{-1} exists.

this also means if an $n \times n$ matrix has n pivots, then it is invertible

from 1.4 we know if an $n \times n$ matrix A has n pivots, then the columns of A span $\mathbb{R}^n$ and the columns are linearly independent

$\rightarrow$ if columns of A are linearly independent, then A^{-1} exists.

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THEOREM 8

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

why $[A \;\; I] \sim \cdots \sim [I \;\; A^{-1}]$ works

$$\left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \overset{rs}{\sim} \cdots$$

is the same as solving $\left[A \;\vdots\; \begin{matrix}1\\0\end{matrix}\right]$ and $\left[A \;\vdots\; \begin{matrix}0\\1\end{matrix}\right]$ at the same time, then put solutions as columns

if solution to $\left[A \;\vdots\; \begin{matrix}1\\0\end{matrix}\right]$ is $\vec{b_1}$

and solution to $\left[A \;\vdots\; \begin{matrix}0\\1\end{matrix}\right]$ is $\vec{b_2}$

then that means $A\vec{b_1} = \begin{bmatrix}1\\0\end{bmatrix}$

and $A\vec{b_2} = \begin{bmatrix}0\\1\end{bmatrix}$

then AB where $B = [\vec{b_1} \;\; \vec{b_2}]$

$= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ and so B must be A^{-1}

Elementary matrix

is a matrix that results when one elementary row operation is performed on an identity matrix

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{\text{elementary matrix}} \quad R_1 + R_3$$

so is $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

all elem. matrices are row equivalent to I so are all invertible