7.4 Derivatives, Integrals, and Multiplications of LTs

last time: $F(s-a) = \mathcal{L}\{e^{at} f(t)\}$

today: $F'(s)$, $\int_s^\infty F(\sigma)\,d\sigma$, $\cancel{F(s)G}$ $F(s)G(s)$

derivative: $F(s) = \int_0^\infty f(t)e^{-st}\,dt$

$$F'(s) = \frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt$$

$$= \int_0^\infty f(t)\frac{d}{ds}e^{-st}\,dt$$

$$= \int_0^\infty f(t)\,e^{-st}\cdot -t\,dt$$

$$= \int_0^\infty [-t f(t)]\,e^{-st}\,dt = \text{LT of } -t f(t)$$

$$\boxed{F'(s) = \mathcal{L}\{-t f(t)\}}$$

or $\mathcal{L}^{-1}\{F'(s)\} = -t\,f(t)$ or $f(t) = \dfrac{\mathcal{L}^{-1}\{F'(s)\}}{-t}$

<u>example</u> $F(s) = \ln\left(\dfrac{s}{s^2-9}\right)$ $\quad f(t) = ?$

definitely not on the Table of common transforms

$$F(s) = \ln(s) - \ln(s^2-9)$$

$$F'(s) = \underbrace{\frac{1}{s}} - \underbrace{\frac{2s}{s^2-9}}$$

we see this in the Table

inv. transform $F'(s)$, then recover $f(t) = \dfrac{\mathcal{L}^{-1}\{F'\}}{-t}$

$$\mathcal{L}^{-1}\{F'\} = 1 - 2\cosh 3t$$

$$f(t) = \frac{\mathcal{L}^{-1}\{F'\}}{-t} = \boxed{-\frac{1}{t}\left(1-2\cosh 3t\right)}$$

$\sinh(t) = \frac{1}{2}(e^{t} - e^{-t})$ $\cosh(t) = \frac{1}{2}(e^{t} + e^{-t})$

$$\mathcal{L}\{\sinh t\} = \frac{1}{s^2 - 1}$$

$$= \frac{1}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1} = \frac{1/2}{s-1} + \frac{-1/2}{s+1}$$

$$= \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}\right\}$$

$$= \frac{1}{2}e^{t} - \frac{1}{2}e^{-t}$$

now $\int_s^\infty F(\sigma)\,d\sigma$

$$F(\sigma) = \int_0^\infty f(t)\,e^{-\sigma t}\,dt$$

$$\begin{aligned}
\int_s^\infty F(\sigma)\,d\sigma &= \int_s^\infty \int_0^\infty f(t)\,e^{-\sigma t}\,dt\,d\sigma \\
&= \int_0^\infty \left(\int_s^\infty f(t)\,e^{-\sigma t}\,d\sigma\right) dt \\
&= \int_0^\infty f(t)\,e^{-\sigma t} \cdot -\frac{1}{t}\Big|_s^\infty dt \\
&= \int_0^\infty f(t)\left(\frac{e^{-st}}{t}\right) dt = \int_0^\infty \left[\frac{f(t)}{t}\right] e^{-st}\,dt
\end{aligned}$$

$$\boxed{\int_s^\infty F(\sigma)\,d\sigma = \mathcal{L}\left\{\frac{f(t)}{t}\right\}}$$

<u>example</u> $\mathcal{L}\left\{\frac{\sin t}{t}\right\}$

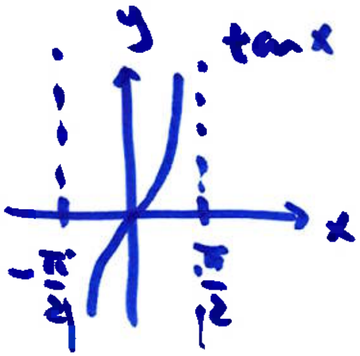
$$\mathcal{L}\left\{\frac{\boxed{f(t)}}{t}\right\} = \int_s^\infty \boxed{F(\sigma)}\, d\sigma$$

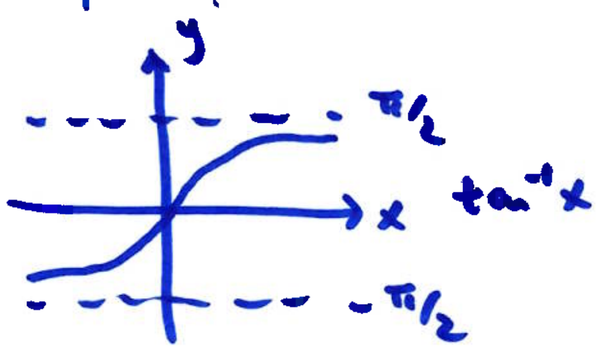
$$\mathcal{L}\{\sin t\} = \boxed{\frac{1}{s^2+1}}$$

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{\sigma^2+1}\, d\sigma$$

$$= \tan^{-1}(\sigma)\Big|_s^\infty = \tan^{-1}(\infty) - \tan^{-1}(s)$$

$$= \boxed{\frac{\pi}{2} - \tan^{-1}(s)}$$





if $\mathcal{L}\{f(t)\} = F(s)$ $\quad \mathcal{L}\{g(t)\} = G(s)$

then $\mathcal{L}^{-1}\{F(s)\,G(s)\} = ?$

work out using definition again, we get

$$\boxed{\begin{aligned} \mathcal{L}^{-1}\{F(s)\,G(s)\} &= \int_0^t f(\tau)\,g(t-\tau)\,d\tau \\ &= \int_0^t f(t-\tau)\,g(\tau)\,d\tau \\ &= f(t) * g(t) \end{aligned}}$$

τ : "tau"
dummy
variables

this is the Convolution Integral of $f(t)$ and $g(t)$

example $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$ one option: $\mathcal{L}^{-1}\left\{\frac{A}{s}+\frac{Bs+C}{s^2+1}\right\}$

or

$$= \mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s}}_{F}\cdot\underbrace{\frac{1}{s^2+1}}_{G}\right\}$$

$F=\frac{1}{s}$ $f(t)=1$

$G=\frac{1}{s^2+1}$ $g(t)=\cancel{\cos t}\ \sin t$

$$= \int_0^t f(t-\tau)\,g(\tau)\,d\tau$$

$$= \int_0^t 1\cdot\cancel{\cos}\ \sin(\tau)\,d\tau = -\cos(\tau)\Big|_0^t = 1-\cos t$$