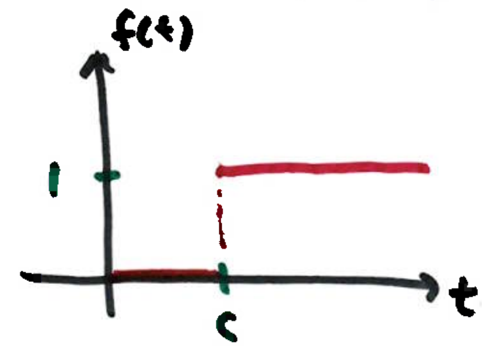
7.5 Piecewise Continuous Input Functions

mass-spring-damper: $mx'' + cx' + kx = f(t)$ ← input

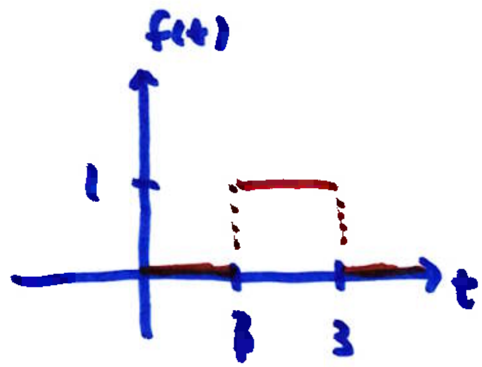
mass, damping, spring

we use a <u>unit step function</u> to model discontinuous $f(t)$

$$u(t-c) = \begin{cases} 1 & \text{if } t \geq c \\ 0 & \text{otherwise} \end{cases}$$



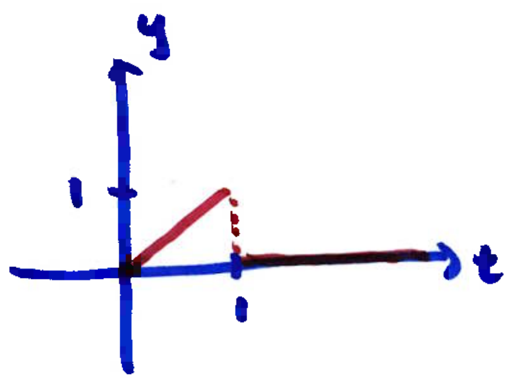
for example,



$$f(t) = \begin{cases} 0 & 0 \leq t < 1 \\ 1 & 1 \leq t < 3 \\ 0 & t \geq 3 \end{cases}$$

$= u(t-1) - u(t-3)$

how about



$$f(t) = \begin{cases} t & 0 \le t \le 1 \\ 0 & t \ge 1 \end{cases} = t + u(t-1)(-t)$$

$$\mathcal{L}\{u(t-c)\} = \int_0^\infty u(t-c)\, e^{-st}\, dt$$

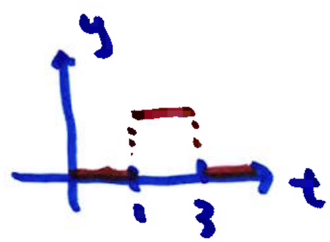
integrate 0 up to $t=c$

integrate 1 from $t=c \to \infty$

$$= \int_c^\infty e^{-st}\, dt = -\frac{1}{s} e^{-st}\Big|_c^\infty = \frac{1}{s} e^{-cs}, \quad s>0$$

$$\boxed{\mathcal{L}\{u(t-c)\} = e^{-cs} \cdot \frac{1}{s}}$$

hat function from earlier :



$$f(t) = u(t-1) - u(t-3)$$

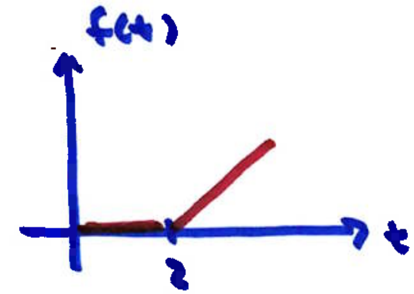
$$F(s) = \mathcal{L}\{u(t-1) - u(t-3)\} = \boxed{e^{-s} \cdot \frac{1}{s} - e^{-3s} \cdot \frac{1}{s}}$$

what about $\mathcal{L}\{u(t-c)\,g(t)\}$? turns out it's NOT $=$ $e^{-cs}\mathcal{L}\{g(t)\}$

for example,

$$f(t) = \begin{cases} 0 & 0 \le t < 2 \\ t-2 & t \ge 2 \end{cases}$$

$$= u(t-2)(t-2)$$



$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\,e^{-st}\,dt$$

$$= \int_2^\infty (t-2)\,e^{-st}\,dt \overset{\text{by parts}}{=} \cdots = e^{-2s}\cdot\boxed{\frac{1}{s^2}}$$

is NOT $\mathcal{L}\{t-2\}$

it is actually is

$\mathcal{L}\{t\}$

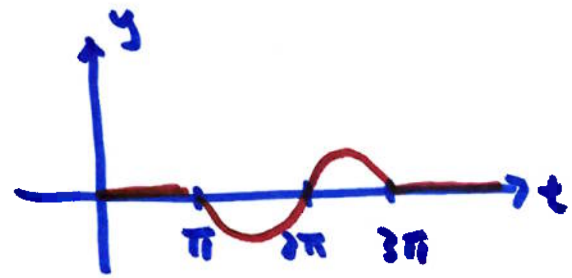
what $t-2$ actually behaves like if it were on at $t=0$

so, $\boxed{\mathcal{L}\{u(t-c)\, f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}}$

what results if we shift the function activated LEFT in t back to origin

example

$$f(t) = \begin{cases} 0 & t \le \pi \\ \sin t & \pi \le t \le 3\pi \\ 0 & t \ge 3\pi \end{cases}$$



$$= u(t-\pi)\sin t + u(t-3\pi)(-\sin t)$$

$$= u(t-\pi)\sin t - u(t-3\pi)\sin t$$

$$\mathcal{L}\{f(t)\} = e^{-\pi s}\mathcal{L}\{\sin(t+\pi)\} - e^{-3\pi s}\mathcal{L}\{\sin(t+3\pi)\}$$

shift LEFT by π means change t to $t+\pi$

$\sin(a+b) = \sin a\cos b + \sin b\cos a$

$$= e^{-\pi s}\mathcal{L}\{-\sin t\} - e^{-3\pi s}\mathcal{L}\{-\sin t\}$$

$$= e^{-\pi s} \cdot -\frac{1}{s^2+1} - e^{-3\pi s} \cdot -\frac{1}{s^2+1}$$

$$= \left(e^{-3\pi s} - e^{-\pi s}\right)\left(\frac{1}{s^2+1}\right)$$

$$\boxed{\mathcal{L}\{u(t-c)\, f(t-c)\} = e^{-cs}\,\underbrace{\mathcal{L}\{f(t)\}}_{\text{transform after changing } t \text{ to } t+c}}$$

inverse transform: do the above in reverse

inverse transform, then shift forward in time (change t to $t-c$)

<u>example</u> $F(s) = e^{-3s}\boxed{\left(\frac{1}{s} + \frac{1}{s+4}\right)}$ → inv. LT ⇒ $1+e^{-4t}$ then change t to $t-3$

$$\boxed{f(t) = u(t-3)\left(1+e^{-4(t-3)}\right)}$$

<u>example</u>

$$F(s) = \underbrace{\frac{1}{s^2}}_{t} - 3e^{-2s}\underbrace{\frac{1}{s^2}}_{t} + 2e^{-3s}\underbrace{\frac{1}{s^2}}_{t}$$

$$f(t) = t - 3u(t-2)\cdot(t-2) + 2u(t-3)\cdot(t-3)$$

Laplace : e^{-cs} for $u(t-c)$, transform function AFTER changing t to $t+c$

Inv. Laplace : $e^{-cs} \to u(t-c)$, inv. LT then change t to $t-c$