9.1 Periodic Functions and Fourier Series (part 1)

Taylor series : $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

if $a=0 \to$ Maclaurin series

Taylor builds $f(x)$ using x^n

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots \qquad -\infty < x < \infty$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \qquad |x| < 1$$

$\frac{1}{2!}$ part of x^2

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

one part of 1

one part of x

basis functions

builds $f(x)$ using linear combos of x^n

Fourier series: same idea, but not using x^n as basis functions

instead, we use $\cos(nt)$ and $\sin(nt)$ $n=0,1,2,3,\ldots$

Taylor: $f(x)$ needs to be <u>analytic</u> near $x=a$

← has derivatives

Fourier: $f(x)$ needs to be periodic on $-\pi \le t \le \pi$ w/ period 2π

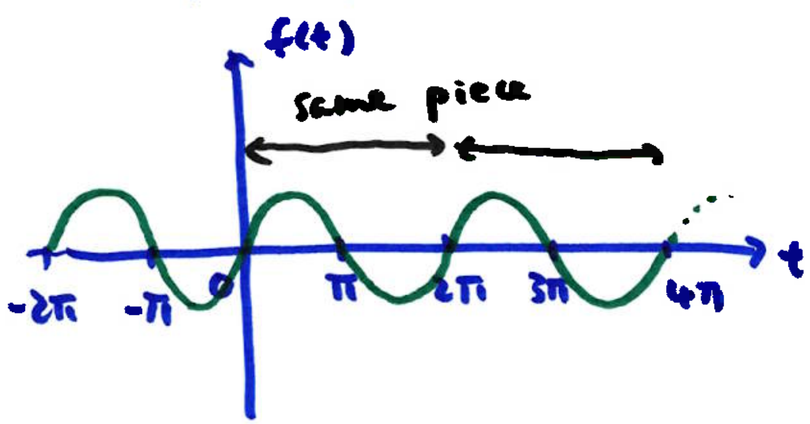
(but that can be adjusted) and <u>piecewise smooth</u>

First, let's define "periodic"

$f(t)$ is periodic with period T if $f(t+T) = f(t)$

for example, $f(t) = \sin(t)$ $T = 2\pi$

$f(0) = f(2\pi)$



if T is the period, then so is $2T$, $3T$, $4T$, etc

the shortest period is called the <u>Fundamental Period</u>

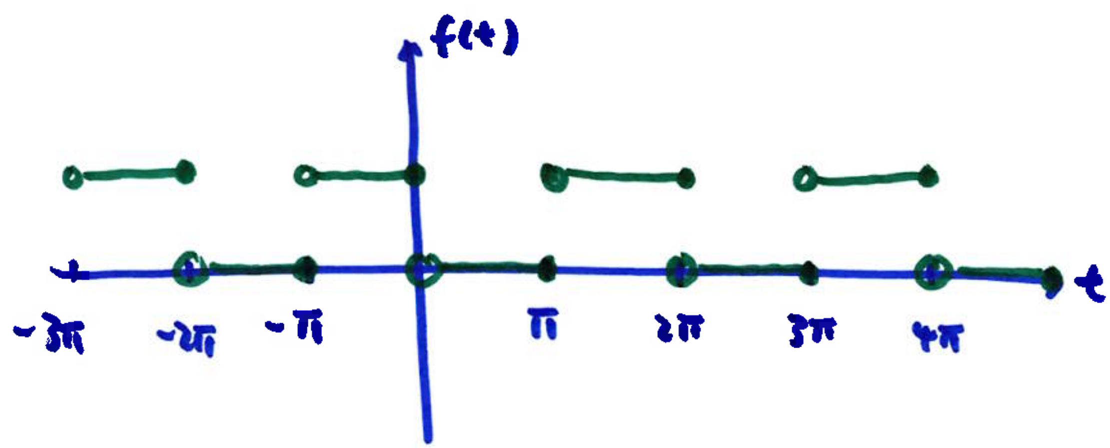
for example, $\sin(t)$ has fundamental period of 2π

$\cos(3t)$ has " " of $\frac{2\pi}{3}$

period is $\frac{2\pi}{3}$

here is another periodic function that repeats w/ $T = 2\pi$

$$f(t) = \begin{cases} 1 & \text{if } -\pi < t \leq 0 \\ 0 & \text{if } 0 < t \leq \pi \end{cases}$$ and repeats every 2π



this is hard to work with due to discontinuities

but Fourier series can approximate it using sines and cosines
(which are easy to work with)

Fourier Series

$f(t)$ periodic w/ period 2π on $-\pi \leq t \leq \pi$

$$f(t) = \frac{1}{2}a_0 + a_1\cos(t) + a_2\cos(2t) + a_3\cos(3t) + \cdots + a_n\cos(nt) + \cdots$$
$$+ b_1\sin(t) + b_2\sin(2t) + b_3\sin(3t) + \cdots + b_n\sin(nt) + \cdots$$

$\cos(0t)$

basis : $\cos(nt)$ and $\sin(nt)$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(nt) + b_n\sin(nt)\right]$$

how to find a_n and b_n ?

$$f(t) = \frac{1}{2}a_0 + a_1\cos(t) + a_2\cos(2t) + a_3\cos(3t) + \cdots + b_1\sin(t) + b_2\sin(2t) + \cdots$$

to find a_n, multiply by $\cos(nt)$ on both sides, then integrate

$\cancel{f(t)}\, a_n =$

$$f(t)\cos(nt) = \frac{1}{2}a_0\cos(nt) + a_1\cos(t)\cos(nt) + a_2\cos(2t)\cos(nt)$$
$$+ \cdots + b_1\sin(t)\cos(nt) + b_2\sin(2t)\cos(nt) + \cdots$$

integrate over $-\pi \le t \le \pi$

$$\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \int_{-\pi}^{\pi}\frac{1}{2}a_0\cos(nt)\,dt + \int_{-\pi}^{\pi}a_1\cos(t)\cos(nt)\,dt$$
$$+ \cdots + \int_{-\pi}^{\pi}b_1\sin(t)\cos(nt)\,dt + \int_{-\pi}^{\pi}b_2\sin(2t)\cos(nt)\,dt$$
$$+ \cdots$$

we know $\displaystyle\int_{-\pi}^{\pi}\cos(at)\cos(bt)\,dt = \begin{cases} \pi & \text{if } a = b \neq 0 \\ 2\pi & \text{if } a = b = 0 \\ 0 & \text{if } a \neq b \end{cases}$

also $\int_{-\pi}^{\pi} \cos(at)\sin(bt)\,dt = 0$

so, it reduces to $\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = a_n \cdot \pi \qquad n = 0, 1, 2, 3, \ldots$

so, $\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt}$

repeat the process by multiplying by $\sin(nt)$ then integrate,

we get $\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt}$ because

$$\int_{-\pi}^{\pi} \sin(at)\sin(bt)\,dt = \begin{cases} \pi & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

<u>example</u>

$$f(t) = \begin{cases} 1 & \text{if } -\pi < t \leq 0 \\ 0 & \text{if } 0 \leq t \leq \pi \end{cases} \quad \text{period of } 2\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt$$

do a_0 first

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\left(\int_{-\pi}^{0} 1\,dt + \int_{0}^{\pi} 0\,dt\right) = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} \cos(nt)\,dt = \frac{1}{\pi}\left(\frac{1}{n}\sin(nt)\right)\Big|_{-\pi}^{0} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{0} \sin(nt)\,dt$$

$$= \frac{1}{\pi}\left(-\frac{1}{n}\cos(nt)\right)\Big|_{-\pi}^{0} = \frac{1}{\pi}\left(-\frac{1}{n} + \frac{1}{n}\underbrace{\cos(n\pi)}_{= \begin{cases} 1 & \text{if } n \text{ even} \\ -1 & \text{if } n \text{ odd} \end{cases}}\right)$$

$$= \frac{1}{\pi}\left(-\frac{1}{n} + \frac{1}{n}(-1)^n\right) = \frac{1}{n\pi}\left(-1 + (-1)^n\right)$$