9.3 Fourier Cosine and Sine Series (part 1)

$f(t)$ is even if $f(-t) = f(t)$ e.g. t^2, t^4, $\cos(t)$

→ y-axis symmetry

$f(t)$ is odd if $f(-t) = -f(t)$ e.g. t, t^3, $\sin(t)$

→ origin symmetry

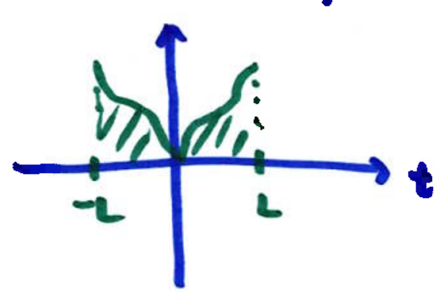
product of two even functions is even

product of two odd functions is even

product of one odd and one even is odd

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(t)\cos\left(\frac{n\pi t}{L}\right)}_{\text{even}} dt$$

if $f(t)$ is even, then $f(t)\cos\left(\frac{n\pi t}{L}\right)$ is even

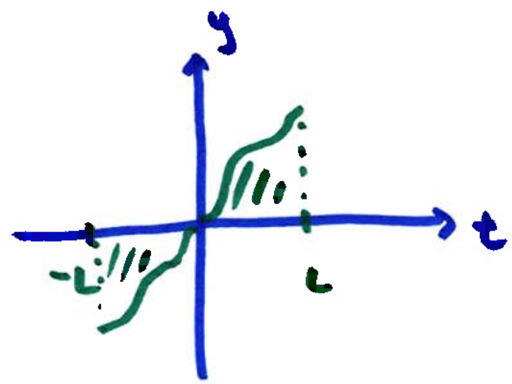


integral → area of shaded region

so, if $f(t)$ is even, then $a_n = \frac{2}{L}\int_{-L}^{0} f(t)\cos\left(\frac{n\pi t}{L}\right)dt$

$$a_n = \frac{2}{L}\int_{0}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

even $f(t)$, $b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(t)}_{\text{even}}\ \underbrace{\sin\left(\frac{n\pi t}{L}\right)}_{\text{odd}}}_{\text{odd}} dt$



net area = 0 so if $f(t)$ is even,

$$b_n = 0$$

if $f(t)$ is odd, following the same process, we find

$$a_n = 0, \quad b_n = \frac{2}{L}\int_{0}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

$f(t)$ <u>even</u> : $a_n = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi t}{L}\right)dt$
$b_n = 0 \rightarrow$ no sine

} Fourier Cosine Series

$f(t)$ <u>odd</u> : $a_n = 0 \rightarrow$ no cosine
$b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right)dt$

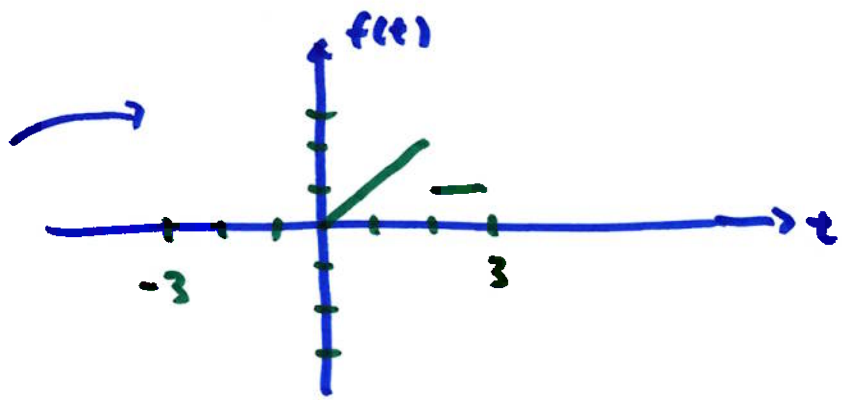
} Fourier ~~Series~~ Sine Series

if $f(t)$ with period $2L$ and given on $0<t<L$ $\rightarrow$ half the picture
to complete the picture, we can choose even or odd extensions

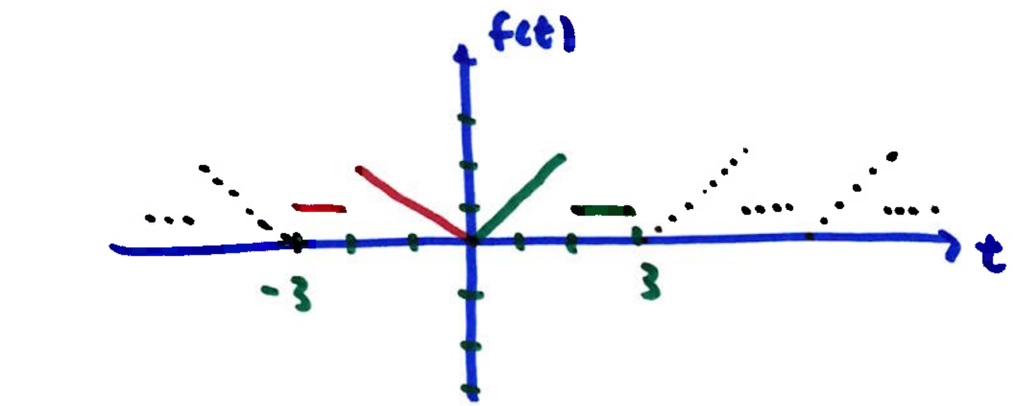
$$f(t) = \begin{cases} t & \text{if } 0<t<2 \\ 1 & \text{if } 2<t<3 \end{cases}$$

$f(t)$ has <u>period 6</u>

?



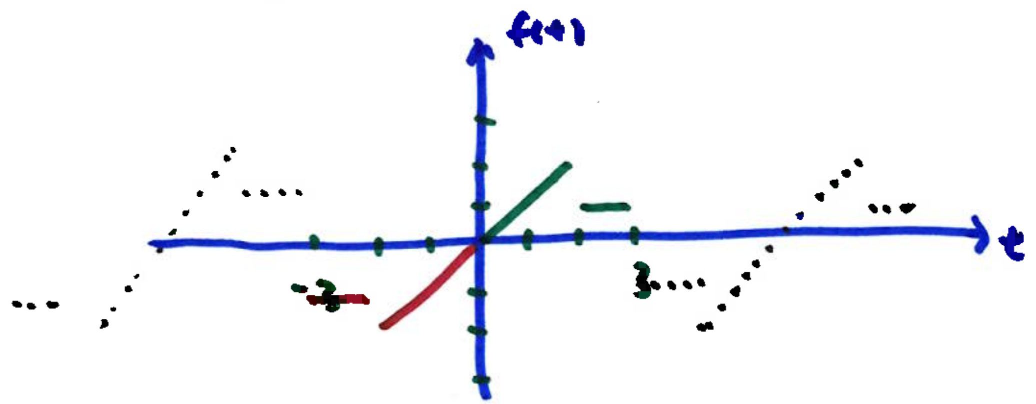
with even extensions → y-axis sym



f(t) is now even

→ Cosine series

with odd extensions



f(t) is now odd

→ Sine series

<u>example</u>

$$f(t) = \begin{cases} t & 0<t<2 \\ 1 & 2<t<3 \end{cases} \qquad \text{period } 6$$

express as a Fourier <u>Sine</u> Series

↳ odd extensions

Sine series → $a_n = 0$

$$b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right)dt \qquad L: \text{half period}$$

$$= \frac{2}{3}\int_0^2 t\sin\left(\frac{n\pi t}{3}\right)dt + \frac{2}{3}\int_2^3 1\cdot\sin\left(\frac{n\pi t}{3}\right)dt$$

$$= \dots = \frac{6\sin\left(\frac{2n\pi}{3}\right) - 4n\pi\cos\left(\frac{2n\pi}{3}\right)}{n^2\pi^2} + \frac{2\cos\left(\frac{2n\pi}{3}\right) - 2\cos(n\pi)}{n\pi}$$

$$\sin\left(\frac{2n\pi}{3}\right) = \begin{cases} \sqrt{3}/2 & n \text{ odd} \\ -\sqrt{3}/2 & n \text{ even} \\ 0 & n = 3k \end{cases} \qquad \cos\left(\frac{2n\pi}{3}\right) = \begin{cases} -\frac{1}{2} & n \neq 3k \\ 1 & n = 3k \end{cases}$$

very difficult to express as summation

→ write a few non-zero terms

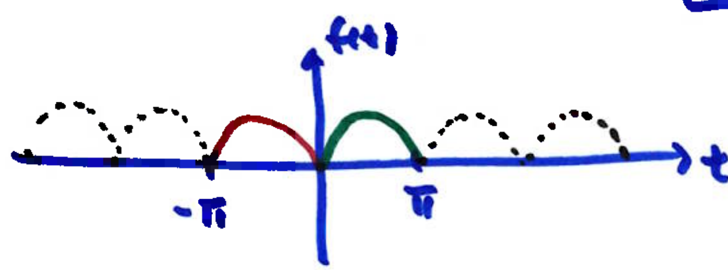
result :

$$f(t) \sim \left(\frac{3\sqrt{3}}{\pi^2} + \frac{3}{\pi}\right)\sin\left(\frac{\pi t}{3}\right) + \left(\frac{-3\sqrt{3}}{4\pi^2} - \frac{1}{\pi}\right)\sin\left(\frac{2\pi t}{3}\right)$$
$$+ \left(\frac{3\sqrt{3}}{16\pi^2} - \frac{1}{4\pi}\right)\sin\left(\frac{4\pi t}{3}\right) + \cdots$$

<u>example</u> $f(t) = \sin(t) \qquad 0 < t < \pi \qquad$ period 2π

express as a Fourier <u>Cosine</u> Series

↳ even extensions



even → $b_n = 0$

$$a_n = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi t}{L}\right)dt \qquad L = \pi$$

$$= \frac{2}{\pi}\int_0^\pi \sin(t)\cos(nt)\,dt$$

$$a_0 = \frac{2}{\pi}\int_0^\pi \sin(t)\,dt = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin(t)\cos(nt)\,dt = \frac{2(\cos(n\pi)+1)}{\pi(1-n^2)} = \begin{cases} \frac{4}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

first few terms

$$\frac{2}{\pi} - \frac{4}{3\pi}\cos(2t) - \frac{4}{15\pi}\cos(4t) - \frac{4}{35\pi}\cos(6t) - \cdots$$

a function doesn't have to be even or odd, it can be neither